Canonicity for Intensional Logics

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This thesis is an account of research undertaken in the Automated Reasoning Project, in the Research School of Information Sciences and Engineering and the Centre for Information Science Research, at the Australian National University between January 1995 and December 1997.

This research was supervised by Dr John Slaney, but unless otherwise indicated, the work presented herein is my own.

None of the work presented here has ever been submitted for any degree at this or any other institution of learning.

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Abstract

A long standing open question asks: "When is an intensional logic determined by its canonical frame?" Certainly the canonical frame will invalidate all non-theses of the logic, however, we can never be sure that the canonical frame itself actually verifies the logic.

This is the starting point for this thesis and we will have in our mind the famous conjecture resulting from KIT FINE's paper [22]:

Conjecture 0.1. A logic is elementary if it is canonical.

While we will not decide this conjecture we will make many points that may be important in its eventual solution.

This thesis starts by introducing our notation and by alluding to the underlying body of research that is used. Apart from a few elementary results the thesis is reasonably self contained and any result that is used yet not specifically derived or cited earlier is either trivial or a well known result of the field of modal logic. All the proofs presented here are worked out in exacting detail and do not presuppose much beyond basic modal logic.

The thesis then moves on to a consideration of neighborhood semantics and discusses the notion of canonicity there. With the thought of how this type of canonicity can be derived without reference to an accessibility relation and its relationship to the more usual sort of canonicity, we look at how we can provide canonicity for a large class of classical logics—the non-iterative logics and a new class of 'even' logics that have the finite model property. One consequence of this work is that the McKinsey logic, recently found to be non-canonical by ROBERT GOLDBLATT [34] and not strongly complete by XIAOPING WANG [101], is neighborhood canonical and so neighborhood strongly complete. The success we experience here will highlight the depth of mystery surrounding the questions of the canonicity of certain iterative logics—these questions still remain open and our techniques cannot be extended. We will highlight a particular example, EK4, and in Appendix A we will indicate some of the results about the nature of the logic that may be helpful in answering the question of its canonicity/finite model property.

Also introduced in this thesis is a novel way of looking at canonicity for normal modal systems without looking at the elementary properties of the canonical accessibility relation. While this technique, known as ultrafilter semantics, does the same job as earlier work such as BJARNI JÒNSSON's [43], it does provide a new and hopefully more intuitive way of looking at the canonicity question. The chapter on ultrafilter semantics is accompanied by a discussion of how it relates to the more general work already present in the literature.

The thesis concludes with two chapters on the relationships between canonical frames. This part of the work is based around the question of how canonical frames over different cardinalities are related and, in turn, this question came from the observation that if the above conjecture held then the following one must certainly hold:

Conjecture 0.2. If the canonical frame for a normal modal logic L in one infinite cardinality satisfies L then the canonical frame in all cardinalities must satisfy L.

Pruning back the problem to this apparently weaker conjecture does not help matters and the relationships between canonical frames examined in the last two chapters are ones that may be important to this conjecture's closure. The first of these two chapters examines the kind of accessibility preserving maps that exist between canonical frames of different cardinality and shows that, when dealing with logics of bounded alternative, we are able to use them to tell a full story. The second of these chapters looks at the types of isomorphisms that exist between canonical frames. It does this by introducing and studying the concept of 'non-standard' automorphisms/isomorphisms and shows that, for some logics, there can be no hope of finding such a map. On the other hand, other logics like the logics of bounded alternative are shown to have no shortage of non-standard isomorphisms and automorphisms.

The thesis is followed by a number of unrelated articles that were written during the author's time as a Ph.D. student at the Australian National University. Included are two papers (one written with KOJI TANAKA and GRAHAM PRIEST) on ADAM GROVE's analysis of two modelings for theory change, and a paper on the use of semantically constrained condensed detachment in automated reasoning (written with JOHN SLANEY).

Contents

Chapter 1

Introduction

1.1 Motivation

This thesis is a projection of a personal journey through the questions raised by the notions of canonicity. As an undergraduate, I was amazed by the way that canonical frames for intensional logics readily gave rise to the completeness of their logics. When presented with a new logical system such as KRISTER SEGERBERG's logic of action [74] and his associated logic of imperatives [75] it was quite simple to provide completeness through the usual methods. Unfortunately the usual methods, while reasonably reliable, did not always work and we had to sometimes consider deeper mechanisms for obtaining completeness, as I found when I considered the canonical structures for the logics of action in my paper [83].

The natural hope then was: If only we understood canonicity better so that when we are given a new logical system we can have a reasonably effective way of determining if a logic is satisfied by its canonical frames.

With this as a hope I started looking into the nature of canonical frames and found that the literature was already populated with results about when logics are canonical and even a result of striking generality about how elementarity implies canonicity—this was a result by KIT FINE [22]. Naturally, I echoed the literature in asking "what about the converse?"

Conjecture 1.1. A canonical logic is elementary.

If we could answer this then we would have a two way bridge between the impervious canonical systems that are used by logicians to prove completeness and the eloquent first order conditions that are used by logicians to draw conclusions about the underlying logic.

This question was obviously a difficult one to answer but then two results turned up which pointed even more to its truth and energised me in my quest to understand these strange structures and, perhaps, answer this question. Both results were due to ROBERT GOLDBLATT, with the first being a result that disqualified a leading candidate counter-example: while JOHAN VAN BENTHEM showed [98] that the McKinsey logic was not elementary GOLDBLATT was able to show [34] that the McKinsey logic was also not canonical. Then in [32] he gave his second result which was that if a logic is determined by an elementary class then the elementary class defined by the canonical frame also determines the logic. Having proven this, GOLDBLATT concludes his book [32] by "placing [a] challenge in front of" his readers: determine whether canonicity implies elementarity.

There is no doubt that this is a difficult problem so the natural way to attack it is to break it down into simpler sub-problems and simpler analyses and to continue to investigate, along GOLDBLATT's lines, the properties of canonicity. Indeed one clear problem due to GOLDBLATT is the question of whether the cardinality of the sets of propositional letters, out of which the canonical frames are constructed, have any relevance to the canonicity of a logic:

Conjecture 1.2. If a normal modal logic L is verified by its canonical frame constructed from $\lambda \geqslant \omega$ many propositional variables, then it is verified by its canonical frame constructed out of κ many propositional variables for all $\kappa > \lambda$.

Clearly if a canonical logic (in some cardinal) is elementary then it is canonical in all cardinals by FINE's result, so this conjecture is in some sense[1] weaker than Conjecture 1.1. Even so, this is still also a difficult problem and must be attacked in little pieces, so, taken by the set theoretic language with which this problem was stated, I wondered if we could use set theoretic tricks to look at the relations between canonical frames of two different cardinalities. While the results along this avenue of investigation still leave many questions open, they do show that interesting things can be said about the relationship between logic, cardinality, and canonicity.

Another avenue for investigation was opened up by my interest in nonnormal logics. While normal logics usually have an accessibility relation associated with them, classical logics are less constrained and are usually thought of in terms of their neighborhood or Montague-Scott semantics. In this type of semantics there is no such thing as *the* canonical frame, but, following BRIAN CHELLAS [13] we are able to define canonicity in terms of what ROY A. BENTON calls [6] *candidate canonical frames*: A classical logic is canonical iff it has a frame that 'conservatively extends' the frame given by the logic and also satisfies the logic. Here we have a conception of canonicity that is independent

[1]While Conjecture 1.1 does not presuppose that we have canonicity in only one infinite cardinal, the tone of the conjecture suggests that the cardinality of the underlying set of propositional letters is irrelevant.

of an accessibility relation and so by investigating this concept we may get a better idea of what is important to canonicity and what is not. My interest in canonicity for classical logics came about because I was asked, by SEGERBERG and CHELLAS, about the canonicity, or the strong completeness, of the classical system EK. This thesis looks at the problems of canonicity for classical systems and produces additional anecdotal evidence for the complexity of the problems associated with canonicity.

Because of these streams of attack on the problems of canonicity, this thesis will only be concerned, on the one hand, with the most general form of classical logics, and, on the other hand, with normal modal logics which have accessibility relations attached to them. As we will see in Section 2.4, this means that we will have multi-modal multi-arity logics for our classical case, and for the normal case we will simply have multi-modal, uni-arity logics. This will exclude from direct consideration a large class of logics for which the modal operators are multi-arity and additive in each coordinate (they distribute over 'and'). At times we will compare our results to those existing in the literature for these additive systems yet we will not consider them as a direct component of our development.

1.2 Thesis Outline

This thesis is divided up so that the initial chapters set the scene and pin down our notation and basic theory. The later chapters each consider a result on canonicity within the two basic thrusts of canonicity for classical logics and cardinality considerations. In the middle of the thesis there is a chapter showing that our approach for neighborhood canonicity can be unified with the normal approach for relational canonicity, at least as far as the Sahlqvist logics are concerned. Appendix A looks at some results on a particular classical logic that, while not solving that logic's canonicity problem, would be of interest to anyone attempting to do so. The other appendices are simply papers which were prepared during the period of my doctoral study and so are included here for completeness.

We will look at each major chapter in turn:

Basic Theory of Boolean Algebras and Intensional Logic

In this chapter we will introduce most of the logical preliminaries that the rest of the work will assume, mostly, without reference. The chapter introduces our basic logical language, it looks at particular logics to which we will occasionally refer, it will discuss the simplification from multi-arity modal operators to uni-arity modal operators for our conception of normal logics, it will introduce the algebraic preliminaries such as boolean and intensional algebras, and it will talk about ultrafilters, both in boolean algebras and over sets in general. Also, the chapter discusses the powerful mathematical technique of ultrapowers.

Relational Semantics

The standard relational semantics for normal modal logics are discussed. This leads to the introduction of the notions of canonicity and the chapter makes clear the history and exact statements of the two large questions which motivated this thesis.

Neighborhood Semantics

This chapter looks at the algebraic semantics for intensional logics and locates within this class a subclass based on powerset boolean algebras. It uses this to define a new semantics, which is then shown to be what is traditionally referred to as the neighborhood semantics. The chapter also goes on to discuss the nature of canonicity for these systems and the relationship between neighborhood and relational frames.

Non-iterative Logics and Canonicity

The question of the canonicity of EK is answered here in full generality: All non-iterative classical logics are shown to be canonical. This tells us that canonicity is natural, and does not depend on any accessibility relation that may be around, at least for these simple logics. We close that chapter by noting that we can find canonical frames for normal systems that are as far away from neighborhood equivalents of relational frames as we can care to imagine.

Canonicity for Even Logics

This chapter introduces a new class of logics. These are the even logics which generalise FINE's uniform normal logics and we show that all even logics with the finite model property are canonical. Again, canonicity is shown to be in some sense natural for these logics. An exciting, yet perhaps disappointing conclusion of this work is that all uniform normal modal logics, and the McKinsey logic is among them, are neighborhood canonical. This is exciting because it is a counterpoint to the results of WANG [101] and GOLDBLATT [34] which show that relational strong completeness and canonicity do not hold, and it is disappointing because it shows that we cannot hope to say too much about relational canonicity by looking at neighborhood canonicity.

Ultrafilter Semantics and Sahlqvist Logics

The results on the McKinsey Axiom not withstanding, it would be nice if we could show that there is a relationship between canonicity proofs given for the classical logics and proofs given for more traditional normal modal logics, in particular the Sahlqvist logics. In this chapter we present a unifying approach to canonicity that can handle the classical and normal logics and can also, in the case of relational semantics, deal with their attendant accessibility relations. While the ultrafilter semantics technique developed here does not manage to go beyond the full swathe of logics that modern algebraic techniques can manage, it does present yet another new way of looking at the canonicity question and it does this by relating it to the 'realisation in a model' approach familiar to logicians.

A Non-standard Injection Between Canonical Frames

This is the first of two chapters which take up the investigation of the relationship between canonical frames formed over sets of propositional letters of different cardinalities. A new type of map, based on ultrafilters and closely aligned to the ultrafilter semantics of the previous chapter, is defined which, while not a frame homomorphism, relates canonical frames of different cardinality. The chapter will devote some effort to showing that every point in the larger frame can be hit by an appropriate map from the smaller frame. It will also point out that this is sometimes the best we can hope for, yet for logics of bounded alternative we get a nice frame homomorphic relationship.

Isomorphisms Between Canonical Frames

The question of the relationship between canonical frames of different cardinalities could be refined to a question about the nature of isomorphisms and automorphisms that may exist between them. This chapter takes up this question and manages to give a full answer for some particular logics and a few hints about what can happen in general.

EK4: A Case Study

This appendix looks at a problem left over from Chapter 5 of whether EK4 is neighborhood canonical. While by the results of Chapter 6 it would follow if EK4 had the finite model property, this thesis cannot answer the canonicity question for EK4. This appendix details some of the results that I have been able to obtain that may be useful to anyone wishing to take up this question.

1.3 Publication History

As is increasingly the custom these days, the bulk of the contents of this thesis have appeared earlier. The chapters which introduce our logical preliminaries have, of course, not appeared before, however parts of them have been taken from the preliminaries sections of the papers which pre-released the later chapters.

Chapter 5 first appeared as an ARP technical report [84] and was later presented at the 1995 Australasian Association for Logic Conference held at the University of New England in Armadale, New South Wales, Australia (and it subsequently appeared as an abstract in *The Bulletin of Symbolic Logic* [91]). After revisions which related the central proof to the wider area of logical endeavour, this paper was accepted for publication in the *Journal of Philosophical Logic* [93]. Due to the more discursive nature of this thesis a section highlighting the extent of the difference between the relational and neighborhood approach to canonicity was added in its appearance here.

Chapter 6 contains the most recent of the papers which combine to form this thesis. It was completed recently when I made one final attempt to solve the problem of the canonicity of EK4. Even though the canonicity of EK4 is still open, the techniques developed here did lead to a solution dependent on EK4 having the finite model property. Also, it did solve a problem about the neighborhood canonicity of the McKinsey Logic which was left open by [93]. This paper has only appeared as an ARP technical report: [92].

Chapter 7 originally appeared spread out over two ARP technical reports, [88] and [87] where the first of these was a general discussion of the importance of neighborhood semantics and the problems around which this thesis is based. It provided an introduction to the conceptually interesting method of "Ultrafilter Semantics" and it showed how this method can be used to prove, in a unified way, both that very simple normal modal logics are canonical and that non-iterative classical logics are canonical. This paper was used as the basis of talks presented at the Australian National University, The University of Liege, and Uppsala University (all in October 1996). The second technical report [87] showed that ultrafilter semantics are able to handle a larger class of logics, namely the Sahlqvist logics, whose relational semantics are characterised by natural elementary conditions. Chapter 7 brings the proof of canonicity for non-iterative and Sahlqvist logics into the one document, making their relationship clearer and also presents the work in the generality of multi-modal logics. Also, the chapter now includes a discussion of the existing literature that deals with the question of proving Sahlqvist's theorem without referring to the underlying accessibility relations.

Chapter 8 first appeared as the technical report [86] and then appeared, without much modification, in the *Journal of the Interest Group in Pure and Applied Logic* [89]. In its incarnation in this thesis some results have been added on the implications of this work to the logics of bounded alternative (originally part of [90], the predecessor of the next chapter, but more at home here). Also, Chapter 8 now presents the work for multi-modal operators.

The technical report [90] was the predecessor of Chapter 9. Apart from the material on homomorphisms between canonical frames for logics of bounded alternative, which was moved to Chapter 8 and the change to multi-modal logics, this chapter essentially follows the form and substance of the original technical report. This work was presented at the conference, *Advances in Modal Logic '96* held at the *Freie Universität Berlin, October 8–10, 1996* and subsequently appeared in the proceedings volume, [94].

Appendix A has not previously appeared. Since this work is only a catalogue of interesting results that may have bearing on the finite model property/canonicity question for EK4, it is unlikely to see print and is only presented here as an aid to any researchers who may want to take up this cause. Again this is to help answer a question first posed in [84] and later asked of a wide audience in the Symbolic Reasoning Systems Workshop III (appearing in its proceedings [85]).

Appendices B, C, and D appeared as the technical reports [65], [95], and [80] respectively. Of these, only the last has been presented at the International Workshop on the Frontiers of Combining Systems in Munich and saw print in the proceedings' volume, [81].

1.4 Mathematical Preliminaries

To appreciate the mathematical content of this work, the reader should have a solid understanding of the basics of ordinal and cardinal arithmetic and an appreciation of basic independence results. An excellent test is to see if you *understand* the statement: '"$2^\omega = 2^{\omega_1}$" is independent of ZFC'. There is absolutely no requirement that the reader be able to derive this, or even understand the mathematics in which this can be derived.

The reader should also have an understanding of basic modal logic with a thorough understanding of the production of canonical models and frames. There is no doubt that an advanced knowledge would be helpful, however no results are imported out of thin air that are beyond the very basic properties of canonical frames—at least not without explicit comment on what the result is and where it comes from.

An excellent reference for basic modal logic is BRIAN CHELLAS's book *Modal Logic: An Introduction* [13], and a good text for advanced mathematical modal logic is MARCUS KRACHT's *Tools and Techniques in Modal Logic* [50]. ROBERT GOLDBLATT's book *Mathematics of Modality* is a significant reference in this work, however all individual results from that work are separately cited. A very advanced survey of modern modal logic can be found in MICHAEL

ZAKHARYASCHEV, FRANK WOLTER and ALEXANDER CHAGROV's book *Advanced Modal Logic* [107].

Having completed the scholastic preliminaries to this thesis, we must now move on to the mathematical ones where we fully ground the theoretical investigations of this work. Let us start our mathematical work by fixing our metalanguage and metatheory.

First we will naturally and without question use the full force of classical logic in our reasoning and all mathematical reasoning will be carried out within the domain of ZFC until the later part of this work when any extra-ZFC assumptions will be clearly noted.

When dealing with simple sets we will try to use upper-case roman letters such as X, Y, Z etc. If the sets are given some structure, such as algebras, we will use typographical conventions which will be laid out when we first encounter a new type of structure. We use $\in$ for the element relation and $\subseteq$ for the subset relation. Ordinals will be denoted by α, β, γ etc., cardinals will be denoted by λ, κ etc., and $\mathrm{card}\,(X)$ will be used to denote the cardinality of the set X.

Fairly often, this work will refer to some results of elementary model theory, and only in extreme cases will they be of sufficient significance or difficulty that they will warrant comment. We will represent first-order formulae with the emboldened greek symbols $\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\chi}$ etc., free variables will be represented by the same variables we use in our proofs (e.g. $x_0, \ldots, x_{n-1}$ or $a_0, \ldots, a_{n-1}$ which usually represent elements in a canonical frame, and elements in a boolean algebra respectively), and we will use a modified turnstyle relation symbol $\models$ to relate a model structure to the formulae it satisfies. Thus

$$(\underline{A}, \underline{I}) \models \boldsymbol{\varphi}$$

for $\boldsymbol{\varphi}$ a first order formula, will mean that the structure $(\underline{A}, \underline{I})$ satisfies, in the first order sense, the first order formula $\boldsymbol{\varphi}$.

In contrast to this, we have the more common (in this thesis) notion of satisfaction $\vDash$ which will mean satisfaction with respect to intensional formulae.

The first order formulae themselves are constructed in the usual way out of function and relation symbols together with the connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\neg$, $\top$, $\bot$ and the quantifiers $\forall$, and $\exists$. We play a trick of 'overloading', and these particular symbols will be overloaded to an almost universal degree with the connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\neg$, $\top$, $\bot$ also used in intensional formulae, and as operations on boolean algebras. While the overloading here may appear excessive, their similar meanings make it natural and it is my expectation that this will significantly diminish confusion.

We do make the exception of using different connectives in our metalanguage. While $\forall$ and $\exists$ will be overloaded to have meaning in our meta-language, we will use $\Longleftrightarrow$, $\Longrightarrow$, and, or, and not to represent our meta-logical notions of 'if and only if', 'implies', 'and', 'or', and 'not'.

The presentation of this dissertation will follow the usual standards adopted by the American Mathematical Society—indeed, *the* standard to which this dissertation is typeset. Theorems, Lemmas, Definitions, Conjectures, etc., will be distinguished in the text and, where relevant, proofs will follow immediately, initiated with the word 'Proof' and ended with the now standard delimiter $\square$. In an effort to make proofs clearer, easy to follow, and for their structure to be transparent, we will highlight the part of a proof we are in by the following typographical conventions:

- **Claimlines**: If we have a metalogical statement we wish to prove, we will often make it clear that this is our goal, underline it and then proceed to provide a proof of this statement. For example

  $\underline{(\forall x, y, z \in A)\,[x = y \text{ and } y = z \Longrightarrow x = z].}$

  Suppose that $x, y, z \in A$, $x = y$, and $y = z$. Then $x = y = z$, so $x = z$.

  The indentation will indicate where a subproof begins and ends.

- **Claims**: Sometimes the proof will be interrupted for a more major claim and these will be indicated as follows

  Claim: Statement of the claim.

  Proof of claim:

  Proof of the claim. *End of proof of claim.*

  Again, the typographical appearance will indicate the stage of the proof that we are in.

- **Cases**: When a proof becomes case ridden it is necessary to indicate exactly which case is being considered. We do this by explicitly highlighting the statement of the case and then carrying out the discussion of that case within an indented environment making it clear where one case starts and the next one ends. Cases will have subcases and in one instance it will have subsubcases. These will all be clearly marked and the level of indentation will make it clear to the reader which case a subcase belongs to and so the stage of the proof that we are in. For example:

  Case First case.

  Discussion of the first case

  Case Second case.

  Discussion of the second case which is broken up into two subcases:

Subcase First subcase

Discussion of the first subcase for the second case.

Subcase Second subcase

...

Subsubcase ...

...

Subsubcase ...

...

- *Inductive Proofs*: In all but the most simple of inductive proofs we will indicate clearly what stage of the inductive proof we are in and make it quite clear what the inductive hypothesis is.

We will use a number of other conventions which, while widely used, are nowhere near universally adopted by practitioners of modern mathematics. These are :

- Sequences of elements of any particular set will be denoted by $\overline{x}$ and we will always assume such a sequence to be finite. By $\overline{x} \in X$ we will mean that each of the terms in the sequence $\overline{x}$ resides within X, however by $\overline{x} \in \overline{X}$ we mean that $x_i \in X_i$ for all $i < \text{length}\,(\overline{x})$. As is usual, if $f : X \longrightarrow Y$ and $\overline{x} \in X$ then we take $f(\overline{x})$ to be the sequence $\overline{y}$ where each $y_i = f(x_i)$. Given two sequences $\overline{x}, \overline{y} \in X$ (not necessarily of the same length), we will use $\overline{x}^\frown\overline{y}$ to denote their concatenation (that is, the sequence $\langle x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}\rangle$ where $n = \text{length}\,(\overline{x})$ and $m = \text{length}\,(\overline{y})$).

- If we have n elements in a set X we will enumerate them as $\langle x_i \mid i < n\rangle$ and so our enumerations will *always* start at 0 and go to $n - 1$ for n the number of items.

- For X a set we will use $\mathcal{P}\,(X)$ to represent the *powerset* of X, that is, the set of subsets of X.

- To denote that a function f has domain W and codomain Z we will write $f : W \longrightarrow Z$. The set of all functions from W to Z will often be written $\{f : W \longrightarrow Z\}$, but sometimes also written ${}^W Z$. If $f \in {}^W Z$ and $Y \subseteq W$ then we use $f\restriction Y$ to represent the restriction of f to Y. If α is an ordinal and Z is a set, we define ${}^{<\alpha}Z$ to be the set $\bigcup\{{}^\beta Z \mid \beta < \alpha\}$.

- If $f : X \longrightarrow Y$ and $Z \subseteq X$ then $f[Z] = \{f(x) \mid x \in Z\}$, the image of Z under f, and if $W \subseteq Y$ then $f^{-1}[W] = \{x \in X \mid f(x) \in W\}$, the inverse image of W.

- The notation ':=' is shorthand for 'equal by definition' and we use it to define objects. For instance the statement

  "We prove that $m := 2n + 1$ is odd."

  can be read as

  "Let $m = 2n + 1$. We prove that m is odd."

Chapter 2

Intensional Logics and Algebras

2.1 Introduction

Before we immerse ourselves in the difficult problems of canonicity let us pause to review the basic theory of intensional logics, their algebraic model structures, and other essential results of mathematics that we exploit in this thesis. This will enable us to pin down our definitions and provide a consistent approach throughout the whole work. Also, the reader should be given a greater sense of where the specialist area of this work departs from the standard theoretical core of intensional logic.

2.2 Propositional Logics and Tautologies

At its most basic level, this work concerns itself with propositional calculi which we consider, after Hilbert, to be constructed out of well formed formulae. Well formed formulae are strings over an alphabet of connectives (of various arity) and primitive elements which we will call propositional variables.

Before we make these concepts exact, let us take a moment to explain the approach we take to propositional variables. In most work on propositional logics, the set P of propositional variables is held fixed as a (usually denumerable) collection of sets so that when they are adjoined into strings using the connectives, no possible confusion will result. Since one of our principle foci of investigation will be how particular structures for the logic change when we admit P of different cardinalities, we must be a little more precise. So throughout this work we will take propositional letters to be ordinals and if P is just given as a "set of propositional variables" we will take this to mean that $P \subseteq$ ORD—the class of all ordinals. In particular, this will allow us to immediately talk about a language over α where α is any ordinal—remember that an ordinal is itself a set of ordinals—and we will often take such an α to be a cardinal.

Whenever not constrained otherwise, we will always take P to be a set of propositional variables. An element of P will usually be denoted by $p, q, r, \ldots$ or $q_0, q_1, q_2, \ldots$. Sometimes we may wish to identify each $\alpha \in P$ with a notational variant $p_\alpha \in P$ which we take to be indistinguishable from α. We make the notational distinction in the hope of avoiding confusion when the ordinal nature of propositional variables is irrelevant to our discussion. We will reserve the right to denote the element $p_\alpha \in P$ by α whenever this seems more suggestive.

In a similar manner we must constrain the notion of connective. We will require a set of connectives to include a few basic ones and beyond that we will not be concerned about their exact number and make up.

Unless otherwise restricted, we take our set of connectives to be:

$$Cnct = \{\wedge, \vee, \rightarrow, \leftrightarrow, \neg, \top, \bot\} \cup \{\Box_i | i \in Idx\}.$$

We do demand that our set of connectives is chosen to ensure that no nonempty string[1] over the alphabet $P \cup Cnct$ will end up back in $P \cup Cnct$.[2]

In addition to this, we will assume that each set of connectives carries with it a function which assigns to each $C \in Cnct$ a natural number $\text{arity}(C)$ representing the arity of the connective C. Moreover, we shall require that the arity function gives unsurprising values to the boolean connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\neg$, $\top$, $\bot$, namely $2, 2, 2, 2, 1, 0, 0$ respectively. Whenever appropriate and when there is no confusion we shall denote $\text{arity}(\Box_i)$ by n_i.

We are now able to define the set of all well formed formulae over the connectives $Cnct$ and the propositional variables P.

Definition 2.1. $\mathcal{S}(Cnct, P)$ is the smallest set satisfying the following conditions:

1. $P \subseteq \mathcal{S}(Cnct, P)$, and
2. if $\varphi, \ldots, \varphi_{n-1} \in \mathcal{S}(Cnct, P)$ and $C \in Cnct$ with $\text{arity}(C) = n$ then

$$\langle C, \varphi_0, \ldots, \varphi_{n-1} \rangle \in \mathcal{S}(Cnct, P).$$

When the set $Cnct$ of connectives is constrained by context we will omit reference to it and write $\mathcal{S}(P)$ for $\mathcal{S}(Cnct, P)$.

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[1] We take a string over an alphabet Σ to simply be a finite sequence over Σ. That is, the strings over Σ are precisely ${}^{<\omega}\Sigma$.

[2] If we did not have this restriction we might end up in the embarrasing situation where the 'formula' $p \wedge q$ is indistinguishable from the propositional variable r. This would cause no end of confusion.

Here we will usually write $\Box_i(\varphi_0, \ldots, \varphi_{n-1})$ (or $\Box_i(\overline{\varphi})$), $\varphi_0 \leftrightarrow \varphi_1$, $\neg\varphi_0$, etc., instead of the more cumbersome $\langle \Box_i, \varphi_0, \ldots, \varphi_{n-1} \rangle$, $\langle \leftrightarrow, \varphi_0, \varphi_1 \rangle$, $\langle \neg, \varphi_0 \rangle$, etc. By $\overline{\varphi} \to \overline{\psi}$ ($\overline{\varphi} \wedge \overline{\psi}$, etc.), we will mean the sequence of formulae constructed by applying the connectives componentwise. That is, the sequence of formulae $\langle \varphi_0 \to \psi_0, \ldots, \varphi_{n-1} \to \psi_{n-1} \rangle$ ($\langle \varphi_0 \wedge \psi_0, \ldots, \varphi_{n-1} \wedge \psi_{n-1} \rangle$, etc.) where $n = \mathrm{dom}(\overline{\varphi}) = \mathrm{dom}(\overline{\psi})$. Also, for unary or nullary connectives we will drop the brackets where unambiguous, as we will for boolean combinations where we assume each $\Box_i$ and $\neg$ bind more strongly than $\wedge$ and $\vee$, and that these in turn bind more strongly than $\to$ and $\leftrightarrow$.

The set $\mathcal{S}(P)$ is called a *language* and its elements are called *well formed formulae* or simply *formulae*. Unless otherwise specified we will take φ, ψ, χ to be such *intensional* formulae.

In a usual way, we are able to import the notion of a *subformula* φ of a formula ψ, which we will write as $\varphi <_{subf} \psi$, and so we also have the notion of recursive definitions over formulae.

Since any well formed formula defines how its component propositional variables are linked up, we can define a substitution in the usual way:

Definition 2.2. Given two sets of propositional variables P and Q we say that σ is a substitution if $\sigma : P \longrightarrow \mathcal{S}(Q)$. We can extend σ to $\widehat{\sigma} : \mathcal{S}(P) \longrightarrow \mathcal{S}(Q)$ in a unique way that respects connectives by requiring that

1. $\widehat{\sigma}(p) = \sigma(p)$ for $p \in P$, and
2. $\widehat{\sigma}(C(\overline{\varphi})) = C(\widehat{\sigma}(\overline{\varphi}))$ for $C \in \mathit{Cnct}$.

Since this extension is unique, we will suppress the "$\widehat{\ }$". If $\varphi \in \mathcal{S}(P)$, $\psi \in \mathcal{S}(Q)$ and $\varphi = \sigma(\psi)$ we say that φ is a *substitution instance* of ψ.

Unless otherwise constrained, we will take σ, τ to be substitutions and, often but not exclusively, we will write $\psi\sigma$ for $\sigma(\psi)$. Given that we wish a substitution σ to satisfy $\sigma(q_j) = \varphi_j$ for $j < n$ we will take $[q_0/\varphi_0, \ldots, q_{n-1}/\varphi_{n-1}]$ to represent the substitution defined as above for $p = q_j$ and by $\sigma(p) = p$ for $p \neq q_1, \ldots, q_n$.

Each formula has a depth of nesting of intensional connectives and we will find it very useful to look at classes of formulae that have bounds on this depth:

Definition 2.3. We define $\mathrm{depth}(\varphi)$, the *depth of a formula* $\varphi \in \mathcal{S}(P)$, as follows:

$$\begin{aligned}
\mathrm{depth}(p) &= 0 \text{ for } p \in P, \\
\mathrm{depth}(\varphi \wedge \psi) &= \max(\{\mathrm{depth}(\varphi), \mathrm{depth}(\psi)\}), \text{ etc., and} \\
\mathrm{depth}(\Box_i \overline{\varphi}) &= 1 + \max(\{\mathrm{depth}(\varphi_j) \mid j < \mathrm{length}(\overline{\varphi})\}).
\end{aligned}$$

Definition 2.4. For P a set of propositional letters and $n \in \omega$ we define $\mathcal{S}_n(P)$, the *collection of formulae of intensional depth at most* n *over* P, as follows:

$$\mathcal{S}_n(P) = \{\varphi \in \mathcal{S}(P) \mid \text{depth}(\varphi) \leqslant n\}.$$

Having fixed our language we are able to introduce the concept of a theory and the notion of a logic itself. Again, we must exercise some care. Most authors define a logic as being a specialised type of subset of a fixed language $\mathcal{S}(P)$ but we need to be more specific as we wish a logic to be an object which captures deductive essence without making a commitment to a particular language.

We do not have this restriction with a theory as it does not necessarily exhibit uniformity, so we can get its definition out of the way.

Definition 2.5. A *theory* over $\mathcal{S}(P)$ is a set $T \subseteq \mathcal{S}(P)$ which is closed under modus ponens, i.e., if φ and $\varphi \to \psi \in T$ then $\psi \in T$.

A logic is a special type of theory, however. As it does not exist with respect to a particular set P, two options present themselves. The first is that we define a logic as a proper subclass of the class of all well formed formulae over all the ordinals, but this is unsatisfying as it shifts a logic to somewhere outside the immediate domain of our set theoretic machinery, so we will adopt the second approach of defining a logic with respect to a specified set, which we choose to be ω. Since a propositional letter in ω could represent any arbitrary propositional letter in ORD, and since logics view propositional letters as stand-ins for arbitrary formulae, we take a logic to be closed under substitution.

Definition 2.6. A logic is a theory L over $\mathcal{S}(\omega)$ which is closed under substitution, i.e., if $\varphi \in L$ and $\sigma : \omega \longrightarrow \mathcal{S}(\omega)$ then $\varphi\sigma \in L$. For any $P \subseteq \text{ORD}$, and $\varphi \in \mathcal{S}(P)$ we say $\vdash_L \varphi$ iff there is a $\sigma : P \longrightarrow \omega$, one-one on the variables in φ, such that $\varphi\sigma \in L$. Any formula φ which satisfies $\vdash_L \varphi$ is said to be a *theorem* or *thesis* of L.

Where convenient we will often write "$\varphi \in L$" when $\varphi \in \mathcal{S}(P) - \mathcal{S}(\omega)$ and $\vdash_L \varphi$. The context of such an expression will clarify the interpretation of "$\in$."

Given this definition, we get the usual definitions of a *consistent* set of formulae, and a *maximal consistent* set of formulae—which, of course, will be maximal with respect to a particular $\mathcal{S}(P)$. Also, let $L(P)$, for L a logic, be the set $\{\varphi \in \mathcal{S}(P) \mid \vdash_L \varphi\}$.

Definition 2.7. If a theory T in the language $\mathcal{S}(P)$ includes $L(P)$ then it is said to be an *L-theory*.

With very little work, we are now able to acquire the standard definitions of inference rules and axiomatisations.

Definition 2.8. An inference rule is a set $\mathcal{R} \subseteq \mathcal{S}\left({}^{n+1}\omega\right)$.

The rule of modus ponens, $\{\langle p, p \rightarrow q, q\rangle\}$ is so central that by definition we require that all logics satisfy it.

Definition 2.9. Let $Ax \subseteq \mathcal{S}(\omega)$ and *Rules* a collection of inference rules, then the logic axiomatised by *Ax* over *Rules* is the smallest set $L \subseteq \mathcal{S}(\omega)$ such that

1. $Ax \subseteq L$,
2. $(\forall \varphi \in L, \sigma : \omega \longrightarrow \mathcal{S}(\omega))\, [\varphi\sigma \in L]$,
3. $(\forall n < \omega)\, (\forall \langle \varphi_0, \ldots, \varphi_n\rangle \in \bigcup \mathit{Rules})\, (\forall \sigma : \omega \longrightarrow \mathcal{S}(\omega))$
   $[(\forall j < n)\, [\varphi_j\sigma \in L] \Longrightarrow \varphi_n\sigma \in L]$, and
4. $(\forall \varphi, \psi \in \mathcal{S}(\omega))\, [\varphi, \varphi \rightarrow \psi \in L \Longrightarrow \psi \in L]$.

Of course, we can then import the usual notions and machinery of proof theory. Also, expressions such as "L is recursively axiomatisable" and "L is recursive", now take on their usual meaning.

We can then introduce, in the usual way, the set *Taut* of all tautologies over $\mathcal{S}(\omega)$ which, we know, is itself a logic and has a recursive set of axioms (over no rules). So if *Ax* is the result of joining a recursive set with *Taut* we end up with a logic which is recursively axiomatisable.

Note that *Taut*, like $\mathcal{S}(\omega)$, is dependent on our choice of *Cnct*, and represents all truth functional tautologies over the set $\mathcal{S}(\omega)$ where $\wedge, \vee, \rightarrow, \leftrightarrow, \neg, \top, \bot$ are taken to have their standard truth functional meaning.

We will tacitly assume that *Taut* is present in all of our axiomatisations.

Definition 2.10. A logic is *equivalential* iff it is closed under the following rule (replacement by provable equivalents):

$$\text{If } \psi[q/\varphi_0] \in L \text{ and } \varphi_0 \leftrightarrow \varphi_1 \in L \text{ then } \psi[q/\varphi_1] \in L.$$

This is a central condition in our logical investigation so we will require that every logic satisfy it. Fortunately it has a nicer form which we refer to as 'the equivalence rule'.

Theorem 2.11. *A logic is equivalential iff for each $i \in Idx$ it is closed under the equivalence rule:*

$$\langle \varphi_0 \leftrightarrow \chi_0, \ldots, \varphi_{n_i-1} \leftrightarrow \chi_{n_i-1}, \Box_i\overline{\varphi} \leftrightarrow \Box_i\overline{\chi}\rangle\,,$$

where $\overline{\varphi}, \overline{\chi} \in \mathcal{S}(\omega)$.

Proof. We prove the *if* and *only if* parts separately.

($\Longrightarrow$) We prove that each component can be replaced by its equivalent and then induction will give that all components can be replaced. Let $i \in Idx$ and let

$$\psi = \Box_i (q, \varphi_1, \ldots, \varphi_{n_i-1}) \leftrightarrow \Box_i (\varphi_0, \varphi_1, \ldots, \varphi_{n_i-1}).$$

Then $\psi[q/\varphi_0]$ is a trivial equivalence so it is in L. Thus by the equivalential property

$$\psi[q/\chi_0] = \Box_i (\chi_0, \varphi_1, \ldots, \varphi_{n_i-1}) \leftrightarrow \Box_i (\varphi_0, \varphi_1, \ldots, \varphi_{n_i-1}) \in L.$$

($\Longleftarrow$) We prove the following by induction on the complexity of ψ, which will give our result.

$$\varphi_0 \leftrightarrow \varphi_1 \in L \Longrightarrow \psi[q/\varphi_0] \leftrightarrow \psi[q/\varphi_1].$$

The base case of this induction is immediate. The Inductive Hypothesis is that the result holds for all $\psi' <_{subf} \psi$ and all φ_0, φ_1, and q. The boolean cases are equally easy. This leaves the intensional case: $\psi = \Box_i \overline{\psi'}$. Now

$$(\forall j < n_i) \left[\psi'_j[q/\varphi_0] \leftrightarrow \psi'_j[q/\varphi_1]\right] \in L$$

by the inductive hypothesis, so the result holds by the hypothesis of the theorem.

$\Box$

We will refer to such logics as intensional logics, a term which comes from the reference to the $\Box_i$ as intensional operators, and the language as an intensional language.

We will take all logics of this work to be equivalential (or "classical" as they are sometimes known), to contain *Taut*,[3] and we will take modus ponens and replacement by provable equivalents as our only rules of inference.

2.3 Particular Logics

Now that we have the notion of a logic we should look at some of the particular logics that are discussed in this thesis. This will set a backdrop for the investigation of this work. Firstly the particular logics discussed in this section will all be *modal logics* which we will take to mean that they are equivalential, *Idx* has cardinality 1, and that $\Box_i = \Box$ is a unary operator with its dual, $\neg\Box\neg$, written as $\Diamond$.

[3]If a logic is axiomatised then the inclusion of *Taut* in the axiom list is tacitly assumed.

The common logics that we will deal with are built up out of collections of axioms. Before we start adding axioms together, it is helpful to have an identifying list of these axioms together with their designators (a bold letter or string of letters to act as a shorthand for the axiom); see Table 2.1.

When naming a logic constructed from these axioms we simply provide a string of designators for the included axioms starting with E which represents the fact that it includes all the tautologies and that it is closed under modus ponens and the equivalence rule. Thus ENKT4 is the equivalential logic with axioms $\Box\top$, $\Box(p \to q) \to (\Box p \to \Box q)$, $\Box p \to p$, $\Box p \to \Box\Box p$ and represents the system more commonly referred to as S4. The logics we form will be analysed with respect to two classes which we tend to deal with individually.

It should be noted here that we could extend our analysis to include logics which are defined using different rules (for example, rule forms of the axioms), however we will restrict ourselves to axiomatisations/logics which have as their only rules, modus ponens and the equivalence rule.

All the logics here are known as *classical logics*, and the simplest classical system is the logic E which is axiomatised by the (classical) tautologies only, hence the name.

A logic which extends ENK is said to be normal and the designators for such logics will start with K and omit E and N. Thus KT45 represents the system ENKT45 which is more commonly known as S5.

This thesis will concern itself with either unrestricted classical logics or the smaller class of normal modal logics.

We will not directly consider logics which include N *but lack* K.

2.4 For Us Normal Logics Are Effectively Unary

Due to the character of the results in the last two chapters of this thesis we will take a rather restricted view of normal modal logics. We could continue our analysis of intensional operators in their greatest generality even when looking at logics which turn out to be based on accessibility relations, however, as most of normal modal logic is carried out with intensional operators of arity one[4] and since the geometric arguments of chapters 8 and 9 will be severely obscured by multi-arity considerations, we will, likewise, restrict our normal modal logics to only have operators of arity one.

To create a clear demarcation line between the logics which we consider to be normal and those which we consider non-normal, we will define normality for logics for arbitrary arity. From this specialised definition we will then

[4]See, e.g., KRACHT'S [50] for an example of a recent survey of modal logic based on polymodal systems whose operators are of arity one.

Designator	Axiom
M	$\Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$
C	$(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$
R	$\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$
K	$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$
N	$\Box\top$
D	$\Box p \rightarrow \Diamond p$
T	$\Box p \rightarrow p$
B	$p \rightarrow \Box\Diamond p$
4	$\Box p \rightarrow \Box\Box p$
5	$\Diamond p \rightarrow \Box\Diamond p$
McK	$\Box\Diamond p \rightarrow \Diamond\Box p$

Table 2.1: Some common axioms and their designators

demonstrate that these logics have intensional operators which can be decomposed into new operators of arity one; that is we will show that these logics are "effectively unary." So in subsequent chapters, by taking a logic to be normal, we will be tacitly assuming that its operators are all of arity one.

Definition 2.12. A logic L is *normal* iff for all $i \in Idx$,

1. $\Box_i (\overline{p} \to \overline{q}) \to (\Box_i \overline{p} \to \Box_i \overline{q}) \in L$ and
2. $\Box_i \overline{\top} \in L$.

where $\overline{p} = \langle p_0, \ldots, p_{n_i - 1}\rangle$, $\overline{q} = \langle q_0, \ldots, q_{n_i-1}\rangle$, and $\overline{\top} = \langle \top, \ldots, \top\rangle$.

Again we must stress that this is not the standard definition that will be found in the literature. For instance, if we translate the definitions used in JÒNSSON and TARSKI's [44] to our setting, we would define a logic L to be normal iff L includes the following formulae, for each $j <$ arity $(\Box_i)$, $\overline{r}_0, \overline{r}_1 \in \mathcal{S}(\omega)$, length $(\overline{r}_0) = j$, length $(\overline{r}_1) =$ arity $(\Box_i) - j - 1$:

1. $\Box_i(\overline{r}_0{}^\frown \langle (p \to q)\rangle {}^\frown \overline{r}_1) \to (\Box_i(\overline{r}_0{}^\frown \langle p\rangle {}^\frown \overline{r}_1) \to \Box_i(\overline{r}_0{}^\frown \langle q\rangle {}^\frown \overline{r}_1))$ and
2. $\Box_i(\overline{r}_0{}^\frown \langle \top\rangle {}^\frown \overline{r}_1)$.

This different view of normality would require that the axiom K 'holds in each component' rather than 'uniformly'. While this approach is much more general and even though it allows for the extensive and sensitive analysis given in [44], it does lumber us with a difficult and largely unfamiliar relational semantics and so this thesis will choose the path of avoiding the complications of true generality.

We will now close this section by showing how our limited and perhaps peculiar notion of normality leads to connectives which are essentially unary.

Theorem 2.13. *$\Box_i (\overline{p} \wedge \overline{q}) \leftrightarrow \Box_i \overline{p} \wedge \Box_i \overline{q}$ is in any normal logic L.*

Proof. First note that $(\overline{p} \wedge \overline{q} \to \overline{p}) \leftrightarrow \overline{\top} \in L$ so by the equivalence rule $\Box_i(\overline{p} \wedge \overline{q} \to \overline{p}) \leftrightarrow \Box_i \overline{\top} \in L$. Thus $\Box_i(\overline{p} \wedge \overline{q} \to \overline{p}) \in L$. Hence by (1) above $\Box_i(\overline{p} \wedge \overline{q}) \to \Box_i \overline{p} \in L$. An analogous argument will then give us that $\Box_i(\overline{p} \wedge \overline{q}) \to \Box_i \overline{q} \in L$, giving $\Box_i(\overline{p} \wedge \overline{q}) \to \Box_i \overline{p} \wedge \Box_i \overline{q} \in L$.

Now, $(\overline{p} \to (\overline{q} \to \overline{p} \wedge \overline{q})) \leftrightarrow \overline{\top} \in L$ which allows us to conclude, (through a derivation sequence which can be summarised as: an application of the equivalence rule, (2), Modus Ponens, (1), and propositional logic twice) that $\Box_i \overline{p} \to (\Box_i \overline{q} \to \Box_i(\overline{p} \wedge \overline{q})) \in L$. But this is just equivalent to $\Box_i \overline{p} \wedge \Box_i \overline{q} \to \Box_i(\overline{p} \wedge \overline{q}) \in L$. This gives us the theorem. □

Definition 2.14. The set of connectives *Cnct* is said to be *unary* iff each $\Box_i \in$ *Cnct* is unary.

Definition 2.15. Given *Idx* the index set of *Cnct* a set of connectives, we set their *unarisations* to be

$$Idx^u = \{\langle i, j\rangle \mid i \in Idx,\ j < n_i\} \text{ and}$$
$$Cnct^u = \{\Box_{\langle i,j\rangle} \mid \langle i, j\rangle \in Idx^u\}$$

respectively. We choose each $\Box_{\langle i,j\rangle}$ to be a new connective distinct from any already in *Cnct* and we take $n_{\langle i,j\rangle} = 1$, i.e., each $\Box_{\langle i,j\rangle}$ is a unary connective.

Definition 2.16. Given *Cnct* a set of connectives define the *translation operator* $\mathrm{tru} : \mathcal{S}\,(Cnct, \mathrm{ORD}) \longrightarrow \mathcal{S}\,(Cnct^u, \mathrm{ORD})$ by

$$\mathrm{tru}\, p = p,$$
$$\mathrm{tru}\,(\varphi \wedge \psi) = \mathrm{tru}\,\varphi \wedge \mathrm{tru}\,\psi, \text{ etc., and}$$
$$\mathrm{tru}\,(\Box_i \overline{\varphi}) = \Box_{\langle i,0\rangle}\, \mathrm{tru}\,\varphi_0 \wedge \cdots \wedge \Box_{\langle i,n_i-1\rangle}\, \mathrm{tru}\,\varphi_{n_i-1}.$$

Definition 2.17. Given a logic L over $\mathcal{S}\,(Cnct, \omega)$ define L^u, the unarisation of L to be the logic with axioms $\{\mathrm{tru}\,\varphi | \varphi \in L\} \cup Taut^u$. Here we take $Taut^u$ to be the set of tautologies in $\mathcal{S}\,(Cnct^u, \omega)$.

Theorem 2.18. *Let L be a logic over $\mathcal{S}\,(Cnct, \omega)$.*

1. *L is a normal logic only if L^u is.*
2. *If L is a normal logic axiomatized by Ax (which includes the normality axioms) then L^u is axiomatised by* $\mathrm{tru}[Ax]$ *and*

$$(\forall \varphi \in \mathcal{S}\,(Cnct, \omega))\,[\varphi \in L \Longleftrightarrow \mathrm{tru}\,\varphi \in L^u].$$

Proof. We prove (1) and (2) separately.

(1) Suppose that L is a normal logic. Now, L^u is an equivalential logic so it just remains to show the following:

$\underline{\Box_{\langle i,j\rangle} \top \in L^u \text{ for all } \langle i, j\rangle \in Idx^u.}$

Let $i \in Idx$, thus $\Box_i\,(\overline{\top}) \in L$. Hence $\mathrm{tru}\,(\Box_i\,(\overline{\top})) = \bigwedge_{j<n_i} \Box_{\langle i,j\rangle} \top \in L^u$ so $(\forall j < n_i)\,[\Box_{\langle i,j\rangle} \top \in L^u]$.

$\underline{\Box_{\langle i,j\rangle}\,(p \to q) \to (\Box_{\langle i,j\rangle} p \to \Box_{\langle i,j\rangle} q) \in L^u \text{ for all } \langle i, j\rangle \in Idx^u.}$

Let $\langle i, j_0\rangle \in Idx^u$. Let $\overline{p} = \langle \top, \ldots, p, \ldots, \top\rangle$, p in the j_0th coordinate. Let $\overline{q} = \langle \top, \ldots, q, \ldots, \top\rangle$, q in the j_0th coordinate. Thus $\Box_i\,(\overline{p} \to \overline{q}) \to$

$(\Box_i\overline{p} \to \Box_i\overline{q}) \in L$. So its uniform translation must be in L^u, that is,

$$\bigwedge_{j<n_i, j\neq j_0} \Box_{\langle i,j\rangle}(\top \to \top) \wedge \Box_{\langle i,j_0\rangle}(p \to q) \wedge \bigwedge_{j<n_i, j\neq j_0} \Box_{\langle i,j\rangle}\top \wedge \Box_{\langle i,j_0\rangle}p$$
$$\to \bigwedge_{j<n_i, j\neq j_0} \Box_{\langle i,j\rangle}\top \wedge \Box_{\langle i,j_0\rangle}q \in L^u.$$

Since $\Box_{\langle i,j\rangle}(\top \to \top)$ and $\Box_{\langle i,j\rangle}\top$ are both in L^u we have that $\Box_{\langle i,j_0\rangle}(p \to q) \wedge \Box_{\langle i,j_0\rangle}p \to \Box_{\langle i,j_0\rangle}q$ as required.

(2) Suppose that L is a normal logic axiomatised by Ax (which include the normality axioms). We wish to show that if $\varphi_1, \ldots, \varphi_k$ is a proof sequence for L over Ax then $\mathrm{tru}\,\varphi_0, \ldots, \mathrm{tru}\,\varphi_{k-1}$ is a proof sequence over $\mathrm{tru}[Ax]$. We proceed by induction on the length k. Suppose the above result holds for k and suppose $\varphi_0, \ldots, \varphi_{k-1}, \varphi_k$ is a proof sequence over $\mathrm{tru}[Ax]$. The formula φ_k can be in the proof sequence for a number of possible reasons, and we examine each reason separately:

Case φ_k is an axiom.

Then $\mathrm{tru}\,\varphi_k \in \mathrm{tru}[Ax]$ and we are done.

Case φ_k is a substitution instance of $\varphi_{k'}$.

But $\mathrm{tru}\,(\varphi_{k'}\sigma) = \mathrm{tru}\,(\varphi_{k'}) \circ \sigma'$ where $\sigma'(p) = \mathrm{tru}\,(\sigma(p))$ and we are done.

Case φ_k arises from $\varphi_{k'} = \varphi_{k''} \to \varphi_k$.

Thus $\mathrm{tru}\,(\varphi_k)$ arises from $\mathrm{tru}\,(\varphi_{k'}) = \mathrm{tru}\,(\varphi_{k''}) \to \mathrm{tru}\,(\varphi_k)$.

Case $\varphi_k = \Box_i\left(\overline{\psi} \leftrightarrow \overline{\chi}\right)$ and $\psi_j \leftrightarrow \chi_j = \varphi_{k_j}$, $k_j < k$, for all $j < n_i$.

Thus $\mathrm{tru}\,(\varphi_{k_j}) = \mathrm{tru}\,(\psi_j) \leftrightarrow \mathrm{tru}\,(\chi_j)$ and so

$$\mathrm{tru}\,(\varphi_k) = \Box_i\left(\mathrm{tru}\,(\overline{\psi}) \leftrightarrow \mathrm{tru}\,(\overline{\chi})\right).$$

This establishes that any valid proof sequence for L can be translated into another valid proof sequence for L^u. $\Box$

Our conclusion from all this is that we can define operators of arity greater than one within systems consisting of operators of arity one.

So, from now on assume that by requiring a logic to be normal we are also requiring all its intensional operators to be unary.

2.5 Boolean Algebras

We now move towards the first type of semantics for our logics. We start by looking at boolean algebras which are the logical semantics for the propositional calculus. Formally:

Definition 2.19. A *boolean algebra* is a structure $\underline{A} = (A, \wedge, \vee, \neg, \top, \bot)$ where A is a set, $\wedge$, $\vee$ are binary operations on A, $\neg$ is a unary operation on A, and $\top, \bot \in A$. Further, the operations and constants satisfy the following algebraic laws:

1	$x \wedge x = x,$	$x \vee x = x,$
2	$x \wedge y = y \wedge x,$	$x \vee y = y \vee x,$
3	$x \wedge (y \wedge z) = (x \wedge y) \wedge z,$	$x \vee (y \vee z) = (x \vee y) \vee z,$
4	$x \wedge (x \vee y) = x \vee (x \wedge y) = x,$	
5	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$
6	$x \wedge \neg x = \bot, x \vee \neg x = \top,$	$\neg\neg x = x,$
7	$\neg(x \wedge y) = \neg x \vee \neg y$, and	$\neg(x \vee y) = \neg x \wedge \neg y.$

Again, we should emphasise that we are overloading the symbols $\wedge$, $\vee$, $\neg$, $\top$, and $\bot$ to have meaning both as connectives in our language and as operators in boolean algebras. We will even go further and suppose that $\rightarrow$ and $\leftrightarrow$ also have direct meanings in boolean algebras. We will indicate that an object is a boolean algebra by placing an underscore, $\underline{\ \ }$, underneath it. Thus $\underline{A}$ is a boolean algebra with an underlying set A and boolean operations which we represent by $\wedge$, $\vee$, $\neg$, $\top$, and $\bot$.

Each boolean algebra has a natural partial order on it which is induced by the boolean operations:

Definition 2.20. For a boolean algebra $\underline{A}$, we define $\leqslant$ and $\geqslant$, relations on A, as follows:

$$a \leqslant b \iff a \wedge b = a \text{ and } a \geqslant b \iff a \vee b = a.$$

Again, we are overloading, and we use context to tell us whether $\leqslant$ represents a numerical or a boolean algebraic comparison. Also, context will always make it clear which boolean algebra the $\leqslant$ comes from. Note that $a \leqslant b \iff \neg b \leqslant \neg a$ and $a \leqslant b \iff b \geqslant a$.

Definition 2.21. Given a set X, we define the *powerset algebra* of X to be the boolean algebra

$$\underline{\mathcal{P}}(X) = (\mathcal{P}(X), \cap, \cup, \sim, X, \varnothing)$$

where $\cap$, $\cup$, and $\sim$ are intersection, union, and X-complementation[5] respectively.

A boolean algebra $\underline{A}$ is called a powerset boolean algebra iff it is isomorphic to $\underline{\mathcal{P}}(X)$ for some set X.

[5]For $Y \in \mathcal{P}(X)$, the *X-complement*, or simply *complement*, of Y is the set $\sim Y := \{x \in X \mid x \notin Y\}$.

To avoid clutter, we will often not use $\sim$ to represent complementation, but rather we overload $\neg$ to take its place, and again, we rely on context to tell us its meaning.

One particular boolean algebra that we will deal with is the two element boolean algebra $\underline{2}$.

Definition 2.22. The two element boolean algebra $\underline{2}$ is the algebra

$$(\{\top, \bot\}, \wedge, \vee, \neg, \top, \bot)$$

where for all $x, y \in \{\top, \bot\}$ $x \wedge y = \top \iff x = y = \top$, $x \vee y = \bot \iff x = y = \bot$, and $\neg x = \top \iff x = \bot$.

If we think of $2 = \{0, 1\}$ as being the underlying set then $\top = 1, \bot = 0$.

There are certain elements which are of special interest in a boolean algebra and these are the elements which behave like singleton sets. Formally:

Definition 2.23. Let $\underline{A}$ be a boolean algebra. Then an *atom* of $\underline{A}$ is an element $a \in A$ which satisfies:

$$(\forall b \in A)\,[b < a \Longrightarrow b = \bot].$$

The collection of all atoms in $\underline{A}$ is denoted by $\mathrm{At}\,\underline{A}$.

So in $\underline{\mathcal{P}}(X)$ each element $\{x\}$, for $x \in X$, is an atom because the only way to reduce the size of $\{x\}$ is to make it empty and the atoms are only of this form since a set of size greater than one can be made non-trivially smaller. Hence X can be put in one-one correspondence with $\mathrm{At}\,\underline{\mathcal{P}}(X)$.

A comprehensive reference for the study of boolean algebras are the books in the *Handbook of Boolean Algebras* series which starts with [48].

2.6 Ultrafilters

Boolean algebras have associated with them the concept of ultrafilters—mathematical objects which are almost ubiquitous in this thesis.

Take $\underline{A} = (A, \wedge, \vee, \neg, \top, \bot)$ to be a fixed boolean algebra.

Definition 2.24. A filter in $\underline{A}$ is a set $\nabla \subseteq A$ such that, for all $a, b \in A$,

1. $a, b \in \nabla \Longrightarrow a \wedge b \in \nabla$, and
2. $a \in \nabla$ and $a \leqslant b \Longrightarrow b \in \nabla$.

A filter ∇ in $\underline{A}$ is called a *proper filter* iff $\nabla \neq A$.

Then there are special types of filters called ultrafilters.

Definition 2.25. A proper filter $\nabla \subseteq A$ is called an *ultrafilter* in $\underline{A}$ iff

$$\nabla \subsetneq A \text{ and } (\forall a \in A)\,[a \in \nabla \text{ or } \neg a \in \nabla]\,.$$

Proposition 2.26. *A filter ∇ in $\underline{A}$ is an ultrafilter iff*

1. *it is* prime, *i.e., if $a \vee b \in \nabla$ then $a \in \nabla$ or $b \in \nabla$, and*

2. $\top \in \nabla, \bot \notin \nabla$.

In order to use ultrafilters we need to be able to generate them, and the following condition is useful in this endeavour.

Definition 2.27. A set $\Sigma \subseteq A$ is said to have the *finite meet property* iff

$$(\forall n < \omega)\,(\forall a_0, \ldots, a_{n-1} \in \Sigma)\,[a_0 \wedge \cdots \wedge a_{n-1} \neq \bot]\,.$$

Proposition 2.28. *If a set $\Sigma \subseteq A$ has the finite meet property then there is an ultrafilter $\nabla \supseteq \Sigma$.*

This gives us a means of constructing ultrafilters: We start with a set that has most of the properties we need and show that it can be extended to an ultrafilter by showing that it has the finite meet property.

When we are just dealing with the term algebra for our logical systems—that is the set $\mathcal{S}(P)$ for some set P—we do not have ultrafilters but we have their natural analogs:

Definition 2.29. Let L be a logic. A set $x \subseteq \mathcal{S}(P)$ is called *maximal L-consistent* (with respect to $\mathcal{S}(P)$) iff it satisfies:

1. The set x is an L-theory, and

2. for all $\varphi \in \mathcal{S}(P)$, either $\varphi \in x$ or $\neg\varphi \in x$.

In Section 2.7 it will become obvious that these maximal consistent sets actually correspond to ultrafilters in the Lindenbaum or canonical algebra.

Ultrafilters are more commonly seen in their set theoretic rather than algebraic setting.

Definition 2.30. By an *ultrafilter over a set Y* we mean an ultrafilter in the boolean algebra $\underline{\mathcal{P}}(Y) = (\mathcal{P}(Y), \cap, \cup, \sim, Y, \varnothing)$.

We use caligraphic uppercase characters (we choose characters around 'U' in the alphabet) to denote this kind of ultrafilter. There will be an ultrafilter extending any set $\Omega \subseteq \mathcal{P}(X)$ iff Ω has the finite intersection property (fip) which is the powerset algebra analog of the finite meet property.

Suppose that we have an ultrafilter $\mathcal{U}$ on a set Y and suppose that $Pr(y)$ is some property of elements of Y. We then write

$$Pr(y) \quad \mathcal{U}\text{ a.e.}$$

to denote that the set $\{y \in Y \mid Pr(y)\}$ is actually in $\mathcal{U}$, and this is read as "$Pr(y)$ holds $\mathcal{U}$ almost everywhere." We often leave out mention of the $\mathcal{U}$ if it is fixed or unambiguous. The following results follow directly from the ultrafilter properties.

Proposition 2.31. *Suppose that Y is a set, that $\mathcal{U}$ is an ultrafilter on Y, and suppose that $Pr_0, \ldots, Pr_{n-1}$ are properties that can be assigned to elements of Y, then the following hold.*

1. *not* $[Pr_1(y)\ \mathcal{U}\text{ a.e.}] \Longleftrightarrow [\text{not } Pr_1(y)]\ \mathcal{U}\text{ a.e.}$,
2. $[Pr_0(y) \text{ or} \cdots \text{or } Pr_{n-1}(y)]\ \mathcal{U}\text{ a.e.} \Longleftrightarrow (\exists i < n)\ [Pr_i(y)\ \mathcal{U}\text{ a.e.}]$, *and*
3. *If* $[Pr_0(y) \Longrightarrow Pr_1(y)]\ \mathcal{U}\text{ a.e.}$ *and* $[Pr_0(y)\ \mathcal{U}\text{ a.e.}]$ *then* $[Pr_1(y)\ \mathcal{U}\text{ a.e.}]$.

We will make use of these properties throughout this work.

2.7 Stone Spaces

STONE proved that every boolean algebra, while not necessarily a powerset boolean algebra, can be thought of as a subalgebra of a powerset boolean algebra over the space of ultrafilters.

Definition 2.32. Let $\underline{A}$ be a boolean algebra. We define the collection of ultrafilters to be

$$\operatorname{ult}(\underline{A}) = \{u \subseteq A \mid u \text{ is an ultrafilter in } \underline{A}\}.$$

While we use ult to identify the ultrafilters over a boolean algebra we will keep this operator conceptually different from the one that induces the topological space of ultrafilters:

Definition 2.33. The *Stone space* of a boolean algebra $\underline{A}$ is the space $X_{\underline{A}} = \operatorname{ult}(\underline{A})$ together with the topology which has as an open basis:

$$\{\|a\| \mid a \in A\} \text{ where } \|a\| = \{x \in X_{\underline{A}} \mid a \in x\}.$$

The study of Stone spaces can get quite involved, as a quick glance through the book *Stone Spaces* [41] will reveal. However, we will restrict ourselves to some basic properties of the Stone space:

Proposition 2.34. *The Stone space of a boolean algebra $\underline{A}$ is a totally separated compact space.*

Totally separated means that any two distinct points are each contained in different members of a disjoint pair of open sets whose union is the whole space.

Any element $\|a\|$ of the basis of the Stone space is both open and closed[6] (because its complement is $\neg\|a\| = \|\neg a\|$ which is in the open basis) and such elements of a topological space are called *clopen*. The basis elements are, in fact, the only sets that are clopen.

Proposition 2.35. *A subset Y of $X_{\underline{A}}$, the Stone space of the boolean algebra $\underline{A}$, is clopen iff it is of the form $\|a\|$ for some $a \in A$.*

Proof. The set Y is open so it is the union of a collection of basic open sets. Say $Y = \bigcup_{a \in E} \|a\|$ for some set $E \subseteq A$. Since Y is also closed, it is a closed subset of a compact space and hence compact. This means that there is an $n \in \omega$ and a collection $\{a_j \mid j < n\} \subseteq E$ such that $Y = \|a_0\| \cup \cdots \cup \|a_{n-1}\|$. This tells us that

$$Y = \|a_0 \vee \cdots \vee a_{n-1}\|$$

as desired. $\square$

We do not often overtly use these properties, rather we mostly see their expression through the behaviour of the underlying boolean algebra, or, more commonly in this investigation, the underlying canonical space.

Definition 2.36. Let L be a logic and P a set of propositional letters. We define the canonical space of L over P to be the set

$$X_P^L = \{x \subseteq \mathcal{S}(P) \mid x \text{ is maximal } L\text{-consistent in } P\}$$

together with the topology which has an open basis

$$\left\{\|\varphi\|_P^L \mid \varphi \in \mathcal{S}(P)\right\}, \text{ where } \|\varphi\|_P^L = \{x \in X_P^L \mid \varphi \in x\}.$$

Where unambiguous we will drop either or both of the superscript L or subscript P.

This is essentially a Stone space, as it is isomorphic to the Stone space for the Lindenbaum algebra of L and so Proposition 2.34 will apply to it also.

[6] A set is *closed* iff its complement is open.

Proposition 2.37. *Suppose that* card $(P) = \kappa \geqslant \max(\{\omega, \text{card}(Idx)\})$. *Then the cardinality of* X_P^L*, for a consistent logic* L*, is* 2^κ.

Proof. By simple cardinal arithmetic we have that

$$\text{card}(\mathcal{S}(P)) = \max(\{\omega, \text{card}(Idx), \kappa\}) = \kappa.$$

Since each $x \in X_P^L$ is a subset of $\mathcal{S}(P)$ we can conclude that card $\left(X_P^L\right) \leqslant 2^\kappa$.

For the reverse inequality note that every $\Delta \subseteq P$ gives rise to a set $\Sigma(\Delta) \subseteq \mathcal{S}(P)$ defined by

$$\Sigma(\Delta) = \Delta \cup \{\neg p \mid p \in P - \Delta\}$$

which is L-consistent because L is itself consistent. The set $\Sigma(\Delta)$ can in turn be expanded into a maximal L-consistent set $y(\Delta)$. The $y : \mathcal{P}(P) \longrightarrow X_P^L$ so defined is then a one-one function and hence $2^\kappa \leqslant$ card $\left(X_P^L\right)$ as desired. □

We do not directly introduce the traditional notion of a Lindenbaum algebra here but rather introduce it as an object constructed out of the canonical space; due to Stone's theorem, this algebra is isomorphic to the algebra obtained through traditional means.

Definition 2.38. The *Lindenbaum boolean algebra* of L is $\underline{A}_P^L$, the boolean algebra which has an underlying set $\{\|\varphi\|_P^L \mid \varphi \in \mathcal{S}(P)\}$ and is considered as a boolean sub-algebra of the *canonical boolean algebra for* L *over* P, $\underline{B}_P^L = \underline{\mathcal{P}}\left(X_P^L\right)$.

Each $x \in X_P^L$ corresponds directly to a member of ult $\left(\underline{A}_P^L\right)$ via the maps $^o : X^L \longrightarrow \text{ult}\left(\underline{A}_P^L\right)$ and its inverse $^\star : \text{ult}\left(\underline{A}_P^L\right) \longrightarrow X^L$, where for $x \in X_P^L$ and $u \in \text{ult}\left(\underline{A}_P^L\right)$,

$$x^o = \{\|\varphi\| \mid \varphi \in x\} \text{ and}$$
$$u^\star = \{\varphi \mid \|\varphi\| \in u\}.$$

On the canonical space for L we have closed sets which are defined, as usual, to be complements of open sets. Since open sets are the unions of basic open sets and since the complement of a basic open set is also basic open we get the following result:

Proposition 2.39. *A set is closed in* X_P^L *iff there is a* $\Sigma \subseteq \mathcal{S}(P)$ *such that the closed set is of the form*

$$\begin{aligned}\|\Sigma\|_P^L &= \bigcap\left\{\|\varphi\|_P^L \mid \varphi \in \Sigma\right\}\\ &= \bigcap_{\varphi\in\Sigma} \|\varphi\|_P^L\end{aligned}$$

where $\Sigma \subseteq \mathcal{S}(P)$.

Each set $\|\Sigma\|_P^L$ can be thought of as representing the theory generated by Σ since the set

$$\left\{\psi \in \mathcal{S}(P) \mid (\forall x \in X_P^L)\left[x \in \|\Sigma\|_P^L \Longrightarrow \psi \in x\right]\right\}$$

is actually $\{\psi \mid \Sigma \vdash_L \psi\}$. So the closed sets in X_P^L are, in effect, the theories of L.

2.8 Intensional Algebras

We now turn to the full notion of algebraic semantics for intensional logics.

Again, let us assume that *Cnct* has already been given.

Definition 2.40. By an *$\mathcal{S}$-algebra,* we mean a structure of the form $\underline{\underline{A}} = (\underline{A}, \underline{I})$ where $\underline{A}$ is a boolean algebra and $\underline{I} = \langle I_i | i \in Idx\rangle$ is a sequence of operators on $\underline{A}$ with arity $(I_i) = $ arity $(\Box_i)$ for each $i \in Idx$.

We will use $\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}}, \ldots$ to represent $\mathcal{S}$-algebras, $\underline{A}, \underline{B}, \underline{C}, \ldots$ for their boolean substructures, and $A, B, C, \ldots$ for their underlying set.

In the event that *Idx* consists of one element we will write $\underline{I}$ for I in the hope of keeping our notation uniform. Each of the I_i is referred to as an interior, and has a dual, $\check{I}_i$, referred to as a closure, defined by $\check{I}_i(\overline{a}) = \neg I(\neg\overline{a})$. Please note that these interiors I_i and closures $\check{I}_i$ are distinct from the interiors and closures induced by the Stone topologies.

We will use lowercase roman $a, b, c, \ldots$ to represent elements of an $\mathcal{S}$-algebra.

These structures are sometimes called expanded boolean algebras (see MARCUS KRACHT's [50, p. 46]). Some specialised forms of these algebras are called boolean algebras with operators [44] and other special forms are called cylindric algebras [38]. When the operations are all unary and the algebra $\underline{\underline{A}}$ satisfies $I_i(\top) = \top$ and $I_i(a \wedge b) = I_i(a) \wedge I_i(b)$ for all $a, b \in A$, we call this a *normal modal algebra*. KRACHT refers to it as a poly-modal algebra [50, p. 47].

We can now power up the whole machinery of universal algebra for use when needed. For the most part though, we will be making use of simple concepts such as homomorphisms (which we will usually denote by $f, g, h, \ldots$), subalgebras (which we will call "$\mathcal{S}$-subalgebras" and where we will write "$\underline{\underline{A}} < \underline{\underline{B}}$" for "$\underline{\underline{A}}$ is an $\mathcal{S}$-subalgebra of $\underline{\underline{B}}$"), products ($\times$ for typographically small products and $\prod$ for typographically large products), free algebras, etc.

Definition 2.41. An algebraic model for $\mathcal{S}(P)$ is a structure of the form $M = (\underline{\underline{A}}, v)$ where $\underline{\underline{A}}$ is an $\mathcal{S}$-algebra and $v : P \longrightarrow A$, called a *valuation.* We define

$\models$, a satisfaction relation between algebraic models and members of $\mathcal{S}(P)$ such that when v is extended to $\mathcal{S}(P)$ by

$$v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi) \text{ etc., and}$$
$$v(\Box_i(\overline{\varphi})) = I_i(v(\overline{\varphi})),$$

we have $(\underline{\underline{A}}, v) \models \varphi \iff v(\varphi) = \top$. We say the theory of M is the set $\mathcal{T}h(M) = \{\varphi \in \mathcal{S}(P) \mid M \models \varphi\}$.

Definition 2.42. We define a satisfaction relation $\models$ between an $\mathcal{S}$-algebra $\underline{\underline{A}}$ and and a member of $\mathcal{S}(P)$ as follows:

$$\underline{\underline{A}} \models \varphi \iff (\forall v : P \longrightarrow A)\left[(\underline{\underline{A}}, v) \models \varphi\right]$$

and we say the logic defined by $\underline{\underline{A}}$ is $\mathcal{L}(\underline{\underline{A}}) = \{\varphi \in \mathcal{S}(\omega) \mid \underline{\underline{A}} \models \varphi\}$.

Notice how $\mathcal{T}h(M)$ and $\mathcal{L}(\underline{\underline{A}})$ satisfy our definitions of a theory and of a logic respectively.

Because $\underline{\underline{A}}$ has a boolean algebra underneath it, we are guaranteed that $\mathcal{L}(\underline{\underline{A}})$ is equivalential. The boolean nature of our algebras also shows us that the formulae of $\mathcal{S}(\omega)$ can be put in a one-one correspondence with particular equations in the first order theory of $\underline{\underline{A}}$ ($\underline{\underline{A}} \models \varphi$ iff the equation $\epsilon(\varphi) = \top$ holds in $\underline{\underline{A}}$ where $\epsilon(p_i) = x_i$ and $\epsilon(\Box_i) = C_i$ etc.) and each equation, corresponds to elements of $\mathcal{S}(\omega)$ ($t_1 = t_2 \longmapsto \epsilon^{-1}(t_1) \leftrightarrow \epsilon^{-1}(t_2)$).

So, we have the following result from either universal algebra or logic.

Theorem 2.43 (Lindenbaum). *Any equivalential logic L has a characteristic algebra $\underline{\underline{A}}$, i.e., $\mathcal{L}(\underline{\underline{A}}) = L$.*

While this, so called, Lindenbaum algebra is important, we follow Section 2.7 and define our notion of a Lindenbaum algebra in terms of the underlying Stone spaces.

Definition 2.44. The *Lindenbaum algebra* of an equivalential logic is the structure $\underline{\underline{A}}_P^L = (\underline{A}_P^L, I^L)$ where $\underline{A}_P^L$ is as given in Definition 2.38, and where I_i^L are operations defined by $I_i^L(\|\overline{\varphi}\|) = \|\Box_i(\overline{\varphi})\|$. The canonical valuation is $v_P^L : P \longrightarrow A$ with $v_P^L(p) = \|p\|$. The canonical model is $M_P^L = (\underline{\underline{A}}_P^L, v_P^L)$.

We note that the equivalential nature of L guarantees that our algebra is well defined.

Again, we stress that this is not the usual Lindenbaum algebra which is constructed out of equivalence classes of formulae, but rather is an algebra isomorphic to it. Because of our future predilection for analysing X_P^L it is a more natural structure to use.

The usual result about canonical models holds, namely that when v_P^L is extended, $v_P^L(\varphi) = \|\varphi\|$ showing that $\mathcal{T}h(M_P^L) = L(P)$.

2.9 Completeness and Decidability

We now look at completeness, the finite model property, and decidability. These concepts will turn out, in places, to be related to the concepts of canonicity—see sections 3.3 and 4.2.

If we are given a class of models M[7] we may ask whether this class both satisfies some logic L and is "enough to fully describe" L, i.e., that it is sound and complete.

Definition 2.45. Let $\mathcal{M}$ be a class of models of the language $\mathcal{S}(\omega)$. We say that $\mathcal{M}$ is *sound* with respect to a logic L iff

$$(\forall M \in \mathcal{M})\,[M \vDash L]$$

and we say that $\mathcal{M}$ is *complete* with respect to L iff

$$(\forall \varphi \in \mathcal{S}(\omega) - L)\,(\exists M \in \mathcal{M})\,[M \nvDash \varphi]\,.$$

If $\mathcal{M}$ is both sound and complete with respect to L we say that $\mathcal{M}$ *characterises* L. We say that L is complete with respect to $\mathcal{M}$ iff $\{M \in \mathcal{M} \mid M \vDash L\}$ characterises L.

In the next chapter we will introduce the first of our specific modelings, or frames, which give rise to a particular subclass of models and we will wonder whether these subclasses are good enough for the logics we have in mind. Thus we need a slightly different notion of completeness.

Definition 2.46. Suppose that $\mathcal{K}$ is a class of structures and suppose that each $F \in \mathcal{K}$ has a class $m(F)$ of models associated with it. We then say that F is a *frame* for a formula $\varphi \in \mathcal{S}(P)$ iff for all $M \in m(F)$ such that M is a model for the language $\mathcal{S}(P)$, $M \vDash \varphi$. We write $F \vDash \varphi$ in this case. We say that $\mathcal{K}$ is *sound* [*complete*] with respect to a logic L iff the class $\bigcup_{F\in\mathcal{K}} m(F)$ is sound [complete] with respect to L. We say that logic L is *complete* with respect to $\mathcal{K}$ iff the class $\{F \in \mathcal{K} \mid F \vDash L\}$ is complete with respect to L.

We can now relate these concepts of soundness and completeness to purely syntactic concepts and considerations.

Definition 2.47. A logic $L \subseteq \mathcal{S}(\omega)$ is *decidable* iff there is an effective procedure for deciding whether an arbitrary $\varphi \in \mathcal{S}(\omega)$ is in or is not in L.

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[7] Here we mean a model over an intensional algebra however it will become clear that models over any of the semantic structures we consider are equivalent.

By effective procedure we mean, of course, a procedure that can be carried out by a Turing machine. This does require that our set of connectives is somehow constrained and for our purposes whenever we refer to a logic as being decidable, we have the sufficiently strong restriction that the set of connectives is finite. If we restrict connectives in this way, we can have a map $\mathrm{gn} : \mathcal{S}(\omega) \longrightarrow \omega$ which *naturally* translates each formula into its Gödel number. Then our definition of decidability becomes:

Definition 2.48. A logic $L \subseteq \mathcal{S}(\omega)$ is *decidable* iff $\mathrm{gn}[L]$ is a recursive subset of ω.

Definition 2.49. A logic L is *recursively axiomatized* iff L is the logic axiomatized by Ax over *Rules*, and both $\mathrm{gn}[Ax] \subseteq \omega$ and $\{\mathrm{gn}(\overline{\varphi}) \mid \overline{\varphi} \in \bigcup Rules\} \subseteq {}^{<\omega}\omega$ are recursive.

Under such conditions $\mathrm{gn}[L]$ is recursively enumerable, since all proofs can be easily enumerated and then recursively checked. Thus, we can demonstrate that a logic is decidable if we can show that $\omega - \mathrm{gn}[L]$ is recursively enumerable and that L has the finite model property.

Definition 2.50. A logic L has the *finite model property* (fmp) iff for each $\varphi \in \mathcal{S}(\omega) - L$ there is a finite model (that is, the underlying algebra is finite) $M \vDash L$ such that $M \nvDash \varphi$.

Note that this is essentially saying that L is complete with respect to the class $\{M \mid M \text{ is finite}\}$.

Theorem 2.51. *If L is recursively axiomatisable, has the fmp, and we can decide if $M \vDash L$ for an arbitrary finite model M then L is decidable.*

Proof. Enumerate all finite models. Decide of each model M whether $M \vDash L$ and if so, start enumerating all the formulae invalid on M. In such a way, we are guaranteed to enumerate all of $\mathcal{S}(\omega) - L$. □

It may be difficult to determine if any arbitrary finite model satisfies L, particularly when the set of axioms is infinite or there are weird rules in *Rules*. However, if we are in a situation where finite counter models can be of a particular type that guarantees that the finite models satisfy the bulk of the axioms and most of the rules, we only have a small amount of work to do to check that $M \vDash L$.

Definition 2.52. A logic $L \subseteq \mathcal{S}(\omega)$ is *finitely axiomatisable* iff it is the logic axiomatised by Ax over the rules of modus ponens and replacement by provable equivalents and $Ax - Taut$ is finite.

Theorem 2.53. *Let L be a finitely axiomatisable logic with the finite model property. Then L is decidable.*

Proof. The logic L is clearly recursively enumerable and checking models now consists of simply checking a finite list of axioms. □

Establishing that certain logics are decidable is no simple matter, however for most common logics it is undertaken by the method of filtrations. In this method, the Stone space is partitioned up into finitely many equivalence classes (usually the equivalence is with respect to a particular finite set of formulae somehow derived from the $\varphi \notin L$ we wish to refute), and it is then shown that the I_i operators can be redefined appropriately.

Using this technique a number of common modal logics can be shown to have the finite model property. In particular, EK, K, S4, S5, KD, KT.

It is important to note that the converse of Theorem 2.53 does not hold: decidability does not imply the finite model property (see e.g., DOV M. GABBAY's [25, pp. 258–265]).

Certain logics are guaranteed to have the finite model property, and this is when they are characterised by a one element class $\{\underline{A}\}$ where A is a finite.

Definition 2.54. Any logic of the form $\mathcal{L}(\underline{A})$ for A finite is called a *tabular* logic.

This name is probably due to the fact that L effectively has a finite table (the algebra) which defines its behaviour. An interesting class of tabular logics is the class of logics which properly extend S5—see SCROGGS's [71].

2.10 Standard Maps

This thesis will make make much of the relationship between different instances of what we will later call canonical frames, but which are for the moment, Stone spaces. "Standard maps" provide a near-immediate relationship between canonical frames over different languages.

Definition 2.55. Suppose that we have a homomorphism $f : \underline{A} \longrightarrow \underline{B}$. Then f induces a map $f_+ : X_{\underline{B}} \longrightarrow X_{\underline{A}}$ defined by:

$$f_+(x) = \{a \in A \mid f(a) \in x\}.$$

Proposition 2.56. *For f and f_+ as given in the previous definition, f_+ really is a map from $X_{\underline{B}}$ to $X_{\underline{A}}$.*

Proof. All we need to show is that $U := \{a \in A \mid f(a) \in x\}$ is an ultrafilter in the boolean algebra $\underline{A}$.

$(\forall a \in U, b \in A)\,[a \leqslant b \Longrightarrow b \in U]$.

Let $a \in U$, $b \in A$ and suppose that $a \leqslant b$. Since f is a homomorphism, $f(a) \leqslant f(b)$, so by $f(a) \in x$ we conclude that $f(b) \in x$ and hence that $b \in U$.

$(\forall a \in A)\,[a \in U \text{ or } \neg a \in U]$.

Let $a \in A$ and assume that $a \notin U$. Thus $f(a) \notin x$ and hence $\neg f(a) \in x$ which tells us that $f(\neg a) \in x$, so $\neg a \in U$.

$\perp \notin U$.

Otherwise $\perp = f(\perp) \in x$ and so x is not a proper filter.

□

Definition 2.57. A map $f : X_{\underline{B}} \longrightarrow X_{\underline{A}}$ is called *standard* iff there is a homomorphism $g : \underline{A} \longrightarrow \underline{B}$ such that $f = g_+$. We say that f is *non-standard* iff it is not standard.

Definition 2.58. We say that a map $f : \mathcal{S}(P) \longrightarrow \mathcal{S}(Q)$ *respects connectives* iff for all $\varphi, \overline{\chi}, \psi \in \mathcal{S}(P)$, it satisfies

1. $f(\perp) = \perp$, $f(\top) = \top$,
2. $f(\varphi \wedge \psi) = f(\varphi) \wedge f(\psi)$, etc., and
3. $(\forall i \in Idx)\,[f(\Box_i \overline{\chi}) = \Box_i f(\overline{\chi})]$.

This is pretty much to say that f is a homomorphism in the term algebra—the free algebra over P without any constraints. Note though, that if $\varphi(\overline{p}) \in L$ then $f(\varphi(\overline{p})) = \varphi(f(\overline{p})) \in L$ by L being closed under substitution.

Suppose that $f : P \longrightarrow \mathcal{S}(Q)$. Then f extends to a connective respecting map which we shall also denote by f. We even use f to represent the map between the canonical algebras defined as follows:[8]

$$f(\|\varphi\|) = \|f(\varphi)\|\,.$$

This induces a map $f_+ : X_Q^L \cong X_{\underline{A}_Q^L} \longrightarrow X_P^L \cong X_{\underline{A}_P^L}$ which can be defined more succinctly by:

$$f_+(x) = \{\varphi \in \mathcal{S}(P) \mid f(\varphi) \in x\}\,.$$

[8]Note that if $\|\varphi\| = \|\psi\|$ then $\varphi \leftrightarrow \psi \in L$ so $f(\varphi) \leftrightarrow f(\psi) \in L$ so $\|f(\varphi)\| = \|f(\psi)\|$.

One consequence of this is that if P and Q are of the same size then there is a bijection f between P and Q which extends to a bijection between $\mathcal{S}(P)$ and $\mathcal{S}(Q)$ and this in turn extends to a bijection between X_Q^L and X_P^L which is really an isomorphism between the Stone spaces.

Conversely if P and Q have different infinite cardinalities and $\mathrm{card}(Idx) \leqslant \mathrm{card}(P)$ then there can be no bijection between $\underline{\underline{A}}_P^L$ and $\underline{\underline{A}}_Q^L$. While this does not mean that there is no Stone space isomorphism between X_P^L and X_Q^L (as we will see in Chapter 9), it does mean that there will be no standard isomorphism between X_P^L and X_Q^L.

2.11 Ultrapowers

In this section we will introduce the notation associated with ultrapowers. This is an important mathematical construction which we use in Chapter 5 to expand algebras so that they 'encompass' powerset boolean algebras.

For this section, take $\underline{\underline{A}} = (\underline{A}, \underline{I})$ be be an $\mathcal{S}$-algebra, take Ω to be a set, and take $\mathcal{U}$ to be an ultrafilter on Ω.

Definition 2.59. The *Ω-power* of $\underline{\underline{A}}$ is the algebra

$$\underline{\underline{A}}^\Omega = \left(\underline{A}^\Omega, \widetilde{\underline{I}}\right)$$

where $A^\Omega = \{\widetilde{a} : \Omega \longrightarrow A\}$ and the operations on A^Ω are defined by

$$\begin{aligned}(\widetilde{a} \wedge \widetilde{e})(\gamma) &= \widetilde{a}(\gamma) \wedge \widetilde{e}(\gamma), \text{ etc., and}\\ \widetilde{I}_i(\overline{\widetilde{a}})(\gamma) &= I_i(\overline{\widetilde{a}(\gamma)}).\end{aligned}$$

Following our usual convention we do not distinguish the boolean operations on A^Ω from the boolean operations on A as the distinction will be clear from context. We use the accent $\widetilde{\ }$ to indicate that we are dealing with elements of a power.

We can use $\mathcal{U}$ to define an equivalence relation $\sim_\mathcal{U}$ on A^Ω by declaring

$$\widetilde{a} \sim_\mathcal{U} \widetilde{e} \Longleftrightarrow \widetilde{a}(\gamma) = \widetilde{e}(\gamma)\ \mathcal{U} \text{ a.e.}$$

We can then use this to get the following, which can be readily seen to be well-defined.

Definition 2.60. The *Ω-$\mathcal{U}$-ultrapower* of $\underline{\underline{A}}$ is the algebra

$$\underline{\underline{A}}^\Omega/U = \left(\underline{A}^\Omega/U, \widehat{\underline{I}}\right)$$

where $A^\Omega/U = A^\Omega/{\sim_\mathcal{U}}$ and the operations on $A^\Omega/\mathcal{U}$ are defined by

$$[\widetilde{a}]_{\sim_\mathcal{U}} \wedge [\widetilde{e}]_{\sim_\mathcal{U}} = [\widetilde{a} \wedge \widetilde{e}]_{\sim_\mathcal{U}}, \text{etc., and}$$
$$\widehat{I}_i([\overline{\widetilde{a}}]_{\sim_\mathcal{U}})(\gamma) = [\widetilde{I}_i(\overline{\widetilde{a}})]_{\sim_\mathcal{U}}$$

Again, the boolean operations are not distinguished and we use the accent $\widehat{}$ to indicate that we are dealing with elements of an ultrapower.

There are particular members of A^Ω that we need to take note of:

Definition 2.61. Define a map $\mathrm{const} : A \longrightarrow A^\Omega$ by setting $\mathrm{const}\,(a)\,(\gamma) = a$.

Definition 2.62. Define a choice function $\mathrm{rep} : A^\Omega/\mathcal{U} \longrightarrow A^\Omega$ by taking $\mathrm{rep}\,(\widehat{a})$ to be any element of A^Ω such that $\widehat{a} = [\mathrm{rep}\,(\widehat{a})]_{\sim_\mathcal{U}}$.

The function const then gives rise to an elementary embedding:

Definition 2.63. The Łos function, $l : \underline{\underline{A}} \longrightarrow \underline{\underline{A}}^\Omega/\mathcal{U}$, is defined by

$$l(a) = [\mathrm{const}\,(a)]_{\sim_\mathcal{U}}.$$

Theorem 2.64 (Łos). *The structure $\underline{\underline{A}}^\Omega/\mathcal{U}$ is well defined and the map $l : \underline{\underline{A}} \longrightarrow \underline{\underline{A}}^\Omega/\mathcal{U}$ is an elementary embedding. That is, for all first order formulae φ and all $\overline{a} \in A$:*

$$\underline{\underline{A}} \models \varphi[\overline{a}] \iff \underline{\underline{A}}^\Omega/\mathcal{U} \models \varphi[l(\overline{a})].$$

Clearly an elementary embedding of one algebra into another is also an injective homomorphism.

We have the following result which could be an immediate consequence of the above theorem had we dealt with many sorted structures.

Lemma 2.65. *Let $u \in \mathrm{ult}\,(\underline{A})$. Then*

$$l(u) = \left\{\widehat{a} \in A^\Omega/\mathcal{U} \mid \mathrm{rep}\,(\widehat{a})\,(\gamma) \in u\,\mathcal{U}\ a.e.\right\} \in \mathrm{ult}\,\left(\underline{A}^\Omega/\mathcal{U}\right).$$

Chapter 3

Relational Semantics

3.1 Introduction

In this chapter we will look at the standard relational semantics for normal modal logics with which most logicians will be familiar. It is because of this familiarity that we will discuss these first and will wait until Chapter 4 before introducing the more general neighborhood semantics which these relational semantics effectively specialise.

The purpose of this chapter is to provide the conceptual and notational basis for the later chapters where canonical relational frames are considered. As a secondary task, this chapter will detail and motivate the main question which inspired this whole work.

This chapter will be concerned with normal multi-modal logics, which means that in all cases throughout this chapter we will be considering languages where $n_i = \text{arity}\,(\Box_i) = 1$ for each $i \in \mathit{Idx}$.

3.2 Relational Frames

The principal structure here is the Relational Frame which is given as follows:

Definition 3.1. A *relational frame* for $\mathcal{S}$ is a structure $(X, \underline{R})$ where X is a non-empty set and, for each $i \in \mathit{Idx}$, R_i is a binary relation on X. If $\text{card}\,(\mathit{Idx}) = 1$ then we just write (X, R).

Since we are dealing with binary relations, we introduce the following piece of notation. Suppose that R is a binary relation on X and $x \in X$ then $R(x)$, the set of all points which x "sees," is defined to be

$$R(x) := \{y \in X \mid \langle x, y\rangle \in R\}$$

and we sometimes refer to this as the set of all R-successors of x.

Each frame induces operations on the algebra of subsets of its underlying space.

Definition 3.2. If $(X, \underline{R})$ is a frame then we say that the $\underline{R}$*-interior* operations induced by $\underline{R}$ consist of functions $I_{R_i} : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ (which we will also denote as $I_{\underline{R}_i}$), for each $i \in Idx$, defined by

$$I_{R_i}(Y) = \{x \in X \mid R_i(x) \subseteq Y\},$$

and their duals $\check{I}_{R_i} : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ (which we will also denote as $\check{I}_{\underline{R}_i}$), for $i \in Idx$, then have definitions

$$\check{I}_{R_i}(Y) = \{x \in X \mid R_i(x) \cap Y \neq \varnothing\}.$$

We can now define a slightly more general structure, which is sometimes called a *general frame* and this will help set up a full duality between these "geometric" objects and the algebras of Chapter 2.

Definition 3.3. A *first order relational frame* for $\mathcal{S}$ is a structure $(X, \underline{R}, \mathbb{P})$ where $(X, \underline{R})$ is a frame and $\mathbb{P}$ a subset of $\mathcal{P}(X)$ which contains X, and is closed under intersection, union, and complement as well as the operations in $\underline{I}_{\underline{R}}$.

We take the frame $(X, \underline{R})$ to be equivalent to the *full* frame $(X, \underline{R}, \mathcal{P}(X))$.

The following definition then makes sense.

Definition 3.4. Suppose that $F = (X, \underline{R}, \mathbb{P})$ is a first order frame then the *dual* of F is the $\mathcal{S}$-algebra

$$F^+ = \big((\mathbb{P}, \cap, \cup, \neg, X, \varnothing), \underline{I}_{\underline{R}}\big).$$

If $F = (X, \underline{R})$ then we take its dual to be the dual of the corresponding full frame:

$$F^+ = (X, \underline{R}, \mathcal{P}(X))^+.$$

Hence we can talk about models and thus satisfaction of formulae.

Definition 3.5. A model on a frame $F = (X, \underline{R}, \mathbb{P})$ (or a full frame $F = (X, \underline{R})$) is a model $M = (F^+, v)$ on the algebra F^+.

We then say that M *satisfies* a formula φ at the point $x \in X$, in symbols $M \vDash_x \varphi$, iff $x \in v(\varphi)$ and we say that M *satisfies* a formula φ, in symbols $M \vDash \varphi$, iff $(\forall x \in X)\,[M \vDash_x \varphi]$. We say that F *satisfies* φ at x, in symbols $F \vDash_x \varphi$, iff for all models M on F, $M \vDash_x \varphi$ and we say that F *satisfies* φ, in symbols $F \vDash \varphi$, iff $(\forall x \in X)\,[F \vDash_x \varphi]$.

For $x \in X$ we say that M *satisfies* the set of formulae Σ at the point $x \in X$, in symbols $M \vDash_x \Sigma$, iff $(\forall \varphi \in \Sigma)\,[M \vDash_x \varphi]$. We define the notions of $M \vDash \Sigma$, $F \vDash_x \Sigma$, and $F \vDash \Sigma$ in an analogous manner.

Note that this is in keeping with Definition 2.46 where we introduced the notion of an arbitrary structure, in this case a relational frame, giving rise to a collection of models. Another thing to note is that each model really does not need complete information about the underlying algebra and in fact, we can have an alternate but equivalent definition of model and satisfaction.

Definition 3.6. A model on a frame $F = (X, \underline{R}, \mathbb{P})$ (or a full frame $F = (X, \underline{R})$) is a triple $M = (X, \underline{R}, v)$ where $v : P \longrightarrow \mathbb{P}$. We then define satisfaction at a point x as follows:

1. $M \vDash_x p \iff x \in v(p)$,

2. $M \vDash_x \varphi \wedge \psi \iff M \vDash_x \varphi$ and $M \vDash_x \psi$, etc., and

3. for each $i \in \mathit{Idx}$, $M \vDash_x \Box_i \varphi \iff (\forall y \in X)\,[\langle x, y\rangle \in R_i \implies M \vDash_y \varphi]$.

The definitions of $M \vDash \varphi$, $F \vDash_x \varphi$, and $F \vDash \varphi$ remain the same.

Note that for $i \in \mathit{Idx}$ our dual connective $\Diamond_i$ has a similar truth condition on models:

$$M \vDash_x \Diamond_i \varphi \iff (\exists y \in X)\,[\langle x, y\rangle \in R_i \text{ and } M \vDash_y \varphi].$$

From these definitions we get the intuition that $\Box_i \varphi$ can be read as "in all possible i-alternatives, φ holds" or more succinctly "φ is i- necessarily true." Dually $\Diamond_i \varphi$ is read as "in some i- possible alternative, φ holds" or more succinctly "φ is i- possibly true."

Definition 3.7. The *theory of a model M* and the *logic of a frame F* are defined as

$$Th(M) = \{\varphi \mid M \vDash \varphi\} \text{ and } \mathcal{L}(F) = \{\varphi \mid F \vDash \varphi\}.$$

Now that we have a functor which assigns a collection of models to each frame (first order and full), we can inherit from Definition 2.46 the notions of soundness and completeness, of a logic being complete with respect to a class, and of L just being complete in a class. One of these is important so let us reiterate it for the special case of relational frames.

Definition 3.8. We say that a logic L is *relational complete* or simply complete iff the following class is complete with respect to L:

$$\{F \mid F \text{ is a full frame and } F \vDash L\}.$$

Such a logic is sometimes referred to as *Kripke Complete* since the relational frames of this chapter are often called *Kripke Frames.*

We did not bother with a notion of first order relational frame completeness since, as we will indicate in the next section, all reasonable logics have this property.

We now have enough sophistication in our semantics to allow us to specialise the concept of completeness:

Definition 3.9. A logic L is *strongly complete* (with respect to the relational semantics) iff for each set of propositional letters P and each L-consistent set $\Sigma \subseteq \mathcal{S}(P)$, there is a frame $F = (X, \underline{R})$, a model M on F, and an $x \in X$ such that $M \models_x \Sigma$.

We often distinguish this notion of strong completeness from ordinary completeness by referring to "ordinary" completeness as "weak" completeness. Strong completeness is sometimes known as compactness.[1]

3.3 Canonical Frames and Canonicity

There is a special type of tool and structure which we use to establish completeness and that is the canonical frame. It is based on the Stone space we introduced in Chapter 2, Section 2.7.

Definition 3.10. Let L be a normal multi-modal logic. We define the *canonical relational frame* over a set of propositional letters P as the frame $F_P^L = \left(X_P^L, \underline{R}_P^L\right)$ where

1. $X_P^L = \{x \subseteq \mathcal{S}(P) \mid x \text{ is a maximal } L\text{-consistent set}\}$, and
2. for $i \in \mathit{Idx}$, $R_{P_i}^L = \left\{\langle x, y\rangle \in {}^2X_P^L \mid (\forall \varphi \in \mathcal{S}(P))\,[\Box_i \varphi \in x \Longrightarrow \varphi \in y]\right\}$.

We then can define the *canonical first order frame* to be $\left(X_P^L, \underline{R}_P^L, \mathbb{P}_P^L\right)$ where[2]

3. $\mathbb{P}_P^L = \{\|\varphi\| \mid \varphi \in \mathcal{S}(P)\}$,

(we are using the $\|.\|$ of Definition 2.33). Then the *canonical model* on the full [on the first order] canonical frame is defined to be $M_P^L = \left(F_P^L, v_P^L\right)$ $[M_P^L = \left(X_P^L, \underline{R}_P^L, \mathbb{P}_P^L, v_P^L\right)]$[3] where

[1]This notion of compactness (see for instance [101]) is *not* the same as the notion of logical compactness ($\Sigma \vdash_L \varphi \Longrightarrow (\exists \text{ finite } \Sigma' \subseteq \Sigma)\,[\Sigma' \vdash_L \varphi]$) that all our logics naturally satisfy.

[2]We do not give this structure a designating name since first order frames will only find use in this chapter.

[3]Strictly speaking, we should introduce a different notational designator for the model on the canonical first order frame but since we will not use it beyond this chapter, and because the effective difference is very slight, we will overload M_P^L.

4. $v_P^L(p) = \|p\|$.

Our definition of canonical model then corresponds to the canonical model on the Lindenbaum algebra (Definition 2.44). Hence, by only having to show that $\|\Box_i\varphi\| = I_{Ri}(\|\varphi\|)$, we get the result:

Proposition 3.11. *Let L be a normal logic and P a set of propositional letters. Then*

$$\mathcal{L}\left(X_P^L, \underline{R}_P^L, \mathbb{P}_P^L\right) = \mathcal{T}h\left(M_P^L\right) = L.$$

This justifies our earlier reticence in talking about a logic being complete with respect to its first order frames; the only logics which would not be so are the non-normal logics and we have excluded them from discussion in this chapter.

Since $\mathcal{T}h\left(M_P^L\right) = L$, we know that $\mathcal{L}\left(F_P^L\right) \subseteq L$, so if we can guarantee that $F_P^L \models L$ we will have demonstrated that $\mathcal{L}\left(F_P^L\right) = L$. In such a case L would be relational complete. Hence this property is important and we give it a name—where we note that it is only the cardinality of P that is important.

Definition 3.12. A normal modal logic L is *relational canonical in cardinal κ* iff $F_\kappa^L \models L$. We say that L is *relational canonical* iff it is relational canonical in all cardinals κ.

3.4 Particular Logics and the Finite Model Property

Canonicity allows us to show that various logics are complete with respect to their relational frames and, more significantly, it allows us to show that some of these logics are complete with respect to special classes of frames.

Remember that all our logics are normal so we can assume that they include the normality axioms.

Example 3.13. By Definition 2.12 the simplest normal modal logic is K_S which is axiomatised by

1. $\Box_i(p \to q) \to (\Box_i p \to \Box_i q) \in L$ for each $i \in \mathit{Idx}$, and

2. $\Box_i \top \in L$ for each $i \in \mathit{Idx}$.

This logic is canonical because its canonical frame, indeed any frame, can be readily seen to satisfy these axioms. This shows that K_S is complete with respect to the class of all frames.

Example 3.14. The mono-modal logic T is axiomatised by the normality axioms together with $\Box p \to p$. The canonical frame F_P^L can be (straightforwardly) shown to have the first order property of R_P^L reflexivity and conversely every frame satisfying this property is also a frame for T. Thus our conclusion is that T is characterised by the class of frames which are reflexive. Moreover it is possible to show that every frame for a normal modal logic L which includes T_i—the axiom T in the ith connective—must satisfy the reflexivity condition on R_i.

Other normal logics such as K4, S4, KD, S5, can be shown to be sound and complete with respect to the class of all "transitive," "reflexive, transitive," "serial,"[4] and "reflexive, transitive, symmetric" frames respectively. Again, the completeness aspect is demonstrated by showing that the canonical frame has the appropriate property.

Now that we have relational semantics we can introduce a special class of logics whose only motivation can be the nature of the underlying relation.

Definition 3.15. For $\overline{\varphi}$ a sequence of formulae of length $n+1$, and $i \in \mathit{Idx}$ we let $\mathrm{Alt}_n^i(\overline{\varphi})$ be the formula

$$\bigwedge_{i \leqslant n} \Diamond_i \varphi_i \to \bigvee_{j<k \leqslant n} \Diamond_i(\varphi_j \wedge \varphi_k).$$

For $\overline{p}$ a sequence of distinct propositional variables, $\mathrm{Alt}_n^i(\overline{p})$ is then the axiom which says that each point can have at most n R_i-successors—this is again proven by showing that any frame for the axiom must satisfy this property, that all frames with this property satisfy the axiom, and that the canonical frame satisfies the axiom. Any logic which contains $\mathrm{Alt}_n^i(\overline{p})$ is called *a logic of bounded alternative in index i*. We say that a logic is simply *of bounded alternative* iff it is of bounded alternative in every index i.

There are even more restrictive logics where we require that each element of a frame has exactly one successor in index i. These are the so called "modal logics with functional alternative relations in index i" and are defined to contain the axiom

$$\Diamond_i p \leftrightarrow \Box_i p.$$

This axiom says that if p holds at one i-successor of a point then it must hold at all i-successors. What happens in the event that the point has no i-successor? Then $\Box_i p$ holds so $\Diamond_i p$ also holds giving lie to the claim that the point had no i-successor. These logics have been comprehensively studied in SEGERBERG's [73].

[4] A relation R is serial iff each point has an R-successor.

Definition 3.16. A relational frame $(X, \underline{R})$ is called *finite* (of size n) iff X is finite (of cardinality n).

Another subclass of the class of logics of bounded alternative is the class of tabular logics. Recall from Definition 2.54 that a tabular logic is one which is $\mathcal{L}(\underline{\underline{A}})$ for some finite $\mathcal{S}$-algebra $\underline{\underline{A}}$. It was an observation of SEGERBERG [72, p. 33] that such an $\underline{\underline{A}}$ is really just a powerset boolean algebra, and through a construction reminiscent of that which produced the canonical frame we get a full finite frame F such that $F^+ = \underline{\underline{A}}$. So we get the following result:

Proposition 3.17. *A logic L is tabular iff there is a full finite relational frame F such that $L = \mathcal{L}(F)$.*

Any tabular logic is clearly of bounded alternative since, for all $i \in Idx$, $\mathrm{Alt}^i_n(\overline{p}) \in \mathcal{L}(F)$ where F is a finite frame of size n.

In actual fact, SEGERBERG's reasoning [72, pp. 31–33] (indicated above for tabular logics) really was reasoning about the finite model property. A simple argument along the same lines will show that every logic with the finite model property really has the finite frame property.

Definition 3.18. A normal logic L has the *finite frame property* iff it is complete with respect to the class of its finite frames.

Proposition 3.19 (Segerberg). *A normal logic L has the finite model property iff it has the finite frame property.*[5]

A lot of everyday logics have been shown to have the finite frame property, including K, T, K4, S4, S5, and KMcK. This last system was shown to have the finite frame property (and hence to be complete and decidable) by KIT FINE in [21]. In most cases, the argument proceeds by taking the canonical frame together with the formula that is to be refuted and using the formula to partition up the canonical frame into a finite number of equivalence classes that can have accessibility relations naturally defined on them. This produces a finite counter frame to the same formula. In Chapter 5 we will use a similar approach to show that non-iterative logics have the finite frame property.

Not all logics have the finite frame property. In [60], DAVID MAKINSON provides a finitely axiomatised logic which does not possess the finite frame property. Some regions of the lattice of logics do have the finite model property throughout, for instance ROBERT BULL has shown [11] that the set of all logics stronger than S4.3[6] has the finite frame property throughout.

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[5] SEGERBERG *did this by showing that each finite model for L is equivalent to one which is differentiated and that a differentiated model really 'is' a full frame for L.*

[6] This is the logic S4 extended with the axiom $\Box(\Box p \to q) \vee \Box(\Box q \to p)$.

There is a class of normal modal logics which is important in the modern study of intensional logics and whose members are often cited as examples/counter- examples:

Definition 3.20. Suppose that $Idx = \{+, -\}$ and that L includes the normality axioms together with

1. $\Box_+ p \to \Box_+\Box_+ p$,

2. $p \to \Box_- \Diamond_+ p$,

3. $p \to \Box_+ \Diamond_- p$,

4. $\Diamond_+ p \land \Diamond_+ q \to \Diamond_+(p \land q) \lor \Diamond_+(p \land \Diamond_+ q) \lor \Diamond_+(q \land \Diamond_+ p)$, and

5. $\Diamond_- p \land \Diamond_- q \to \Diamond_-(p \land q) \lor \Diamond_-(p \land \Diamond_- q) \lor \Diamond_-(q \land \Diamond_- p)$.

Then L is said to be a *tense logic*.

The intuition here is that $\Box_+\varphi$ should be read as "φ holds throughout the future," and $\Box_-\varphi$ should be read as "φ holds throughout the past." Axioms 2 and 3 essentially require that R_+ is the relation converse of R_- and axioms 1, 4, and 5 represent the requirement that these relations are linear.

Another thing worth mentioning at this point is that for certain purposes we can dispense with the possibility that $\mathrm{card}(Idx) > 1$. It was shown by KRACHT and WOLTER in [51] that multi-modal normal logics can be *simulated* by mono-modal logics in the sense that the many modal operators of the multi-modal logic can be defined in terms of the single modal operator of a mono-modal logic. Also, the mono-modal logic can be axiomatised in such a way that a formula is a theorem of the multi-modal logic precisely when its translation is a theorem of the mono-modal logic. When this is done, it turns out that properties like decidability, finite model property, tabularity, canonicity, and many more hold of the multi-modal logic exactly when they hold of the corresponding mono-modal logic.

A class that also crops up is the class of subframe logics. These logics, introduced by FINE in [23], are normal multi-modal logics whose collection of (full) relational frames is closed under the taking of subframes:

Definition 3.21. Suppose that $F = (X, \underline{R})$ and that $Y \subseteq X$. Then the subframe of F induced on Y is the frame $(Y, \underline{R'})$, where for each $i \in Idx$, $R'_i = R_i \cap (Y \times Y)$.

It turns out that subframe logics which are extensions of K4 have nice properties, one of which is the finite model property.

3.5 Canonicity and Elementarity

The central question of this thesis is "which logics are canonical and how does canonicity relate to other properties of a logic?" At its simplest level we look at properties of the axioms itself, such as whether they are non-iterative or even, and we move on to more complicated properties such as whether they are determined by elementary classes. In this section we will make some of these more complicated notions precise.

The first notion is that of a descriptive frame which gives rise to the notion of d-persistence which turns out to be equivalent to canonicity.

Definition 3.22. A first order frame $F = (X, \underline{R}, \mathbb{P})$ is called *descriptive* iff

1. for all $x, y \in X$ with $x \neq y$, there is a $Y \in \mathbb{P}$ such that $x \in Y$ and $y \notin Y$,
2. for all $x, y \in X$ and $i \in Idx$,
$$(\forall Y \in \mathbb{P})\,[x \in I_i(Y) \Longrightarrow y \in Y] \Longrightarrow \langle x, y\rangle \in R_i, \text{ and}$$
3. every proper filter $\nabla \subseteq \mathbb{P}$ satisfies $\bigcap \nabla \neq \varnothing$.

Clearly the canonical first order frame for a logic is descriptive and a logic L is canonical if its 'descriptive' first order frame is a frame for L when stripped of $\mathbb{P}^L_P$, its set of propositions.

Definition 3.23. A logic L is *d-persistent* iff every descriptive first order frame $F = (X, \underline{R}, \mathbb{P})$ for L satisfies $(X, \underline{R}) \vDash L$.

Through a straightforward argument which embeds each frame into a canonical frame over a suitably large cardinal and which relies on the concept of a generated subframe (which we will introduce in Definition 3.42) we can get the following result:

Theorem 3.24. *A logic is canonical iff it is d-persistent.*

Another concept in the realm of canonicity is that of complexity.

Definition 3.25. A logic L is *relational complex* iff each algebra for L can be homomorphically embedded into F^+ for some full frame F.

We noted that logics like K, T, S4 have collections of frames which are defined by first order conditions such as "no conditions," "reflexivity," and "quasi order." (A quasi order is a reflexive transitive relation.) It turns out that a lot of logics are in this category.

Definition 3.26. An *elementary class* of frames is a class of frames defined by a collection of first order formulae over the language of equality and binary relations, i.e., for Σ a set of first order formulae, it is a class of the form

$$\{(X, \underline{R}) \mid (X, \underline{R}) \models \Sigma\}.$$

A logic is said to be elementary iff it is characterised by some elementary class.

Note that a logic being elementary does not mean that all its frames are in an elementary class, just that every non-thesis of the logic is refuted by some frame in that elementary class and that the class universally gives rise to the logic.

In 1975 HENRIK SAHLQVIST [68] came up with a class of modal formulae which were naturally equivalent to first order conditions in that the collection of frames satisfying one of these modal formulae also satisfied an elementary condition derivable from that modal formula, and vice versa. This meant that a large class of logics were immediately amenable to analysis just in terms of their first order 'equivalents', rather than through the essentially second order definition of truth and satisfaction in modal models. We will not define the class of Sahlqvist formulae in this chapter, but we will discuss them in depth in Chapter 7 where we recreate an important consequence of SAHLQVIST's work.

Sahlqvist formulae, of course, define Sahlqvist logics and important elements of this class are the logics we have met like K, T, $\mathsf{D4}$, etc. But then there is a logic like KMcK which is one of the simplest logics that is not Sahlqvist. JOHAN VAN BENTHEM [98] has shown that this logic is not elementary, so not all logics are elementary. An interesting result proven by KIT FINE [22] gives us a 'test' for elementarity.

Theorem 3.27 (Fine). *If a logic is elementary then it is canonical.*

It is natural to wonder about the converse of this theorem. Is it possible that there is a canonical logic which is not elementary? A natural test of this is the logic KMcK which is not elementary. It took a long time but eventually ROBERT GOLDBLATT showed in [34] that this logic is not canonical. Note though, that failure of canonicity does not imply the failure of completeness as KMcK was shown to have the finite frame property and hence to be complete with respect to its class of finite frames.

This leads to the big conjecture which inspired this thesis and which so far remains open:

Conjecture 3.28. A normal modal logic is canonical only if it is elementary.

This conjecture seems quite reasonable and as yet there is no answer. Compelling anecdotal evidence for this conjecture has been provided by GOLDBLATT who has shown [32]:

Theorem 3.29 (Goldblatt). *If a normal modal logic L is elementary then it is characterised by the elementary class*[7]

$$\{(X, \underline{R}) \mid (\forall\varphi)\ [(X_P^L \underline{R}_P^L) \models \varphi \Longrightarrow (X, \underline{R}) \models \varphi]\}.$$

This means that if we want to know which elementary formulae 'define' a particular elementary modal logic we need only inspect the canonical frame of that logic. Of course, this collection of first order formulae will have a lot of unnecessary chaff in it. For instance the canonical frame for the elementary logic T is infinite so in addition to reflexivity, the only really crucial elementary condition, our inspection of the canonical frame would also yield unnecessary conditions such as, for each $n \in \omega$, "there are at least n elements."

Unfortunately, this result requires that we start with a logic that we know to be elementary, so this result does not close the conjecture.

Another piece of anecdotal evidence was obtained by FRANK WOLTER in [103]:

Theorem 3.30 (Wolter). *Every canonical tense logic is elementary.*

Other relationships abound. For instance it is clear that if L is canonical then it is strongly complete. Also, WOLTER has shown [102] that for normal modal logics, strong completeness is equivalent to complexity.

To summarise, for L a normal modal logic, we have the following relationships between the properties discussed so far.

$$\begin{array}{c}
L \text{ is elementary} \\
?\Uparrow? \qquad\qquad \Downarrow \\
L \text{ is canonical } (\Longleftrightarrow L \text{ is d-persistent}) \\
\Downarrow \\
L \text{ is strongly complete } (\Longleftrightarrow L \text{ is relational complex}) \\
\Downarrow \\
L \text{ is complete}
\end{array}$$

[7] *Remember that P is an infinite set of propositional letters.*

3.6 Morphisms and Duality

In this section we will introduce the essential duality between relational frames and the boolean algebras with operators set up in the previous section. This work has a long history. Longer perhaps than the purely relational analysis of modal logic which was essentially started by SAUL KRIPKE's landmark paper [54] in 1963. BJARNI JÓNSSON and ALFRED TARSKI produced a paper, [44] which, in some sense, anticipated the relational semantics of KRIPKE. In that paper they started with boolean algebras with operators, (which in their formalism required that the algebras be additive—something the algebras for normal modal logics naturally satisfy) and they showed how to move back and forth between these algebras and 'relation' structures which really were the relational frames first introduced by KRIPKE and are the frames of this chapter. This work was later re-explored in part by LEMMON [56, 57] who also investigated the duality presented here.

We saw that a relational frame naturally gives rise to a modal algebra and it was on this algebra that we based our truth definitions. (We went on to specialise the truth definition to obtain truth at a point in a model.) This provided the first half of the duality which we have already seen as Definition 3.4:

Definition 3.31. Suppose that $F = (X, \underline{R}, \mathbb{P})$. Then the *algebraic dual of* F, or F^{+}, is the algebra $(\underline{A}, \underline{I})$ where $\underline{A}$ is the boolean set algebra with field $\mathbb{P}$, and $\underline{I} = \underline{I}_{\underline{R}}$. In the event that F is a full frame, F^{+} is based on the boolean algebra $\underline{\mathcal{P}}(X)$.

So turning a relational frame into an algebra is easy. The conversion in the other direction requires just a little more sophistication and it is essentially the construction of the canonical frame:

Definition 3.32. Suppose that $\underline{\underline{A}} = (\underline{A}, \underline{I})$ is an algebra which gives rise to a normal modal logic. Then $\underline{\underline{A}}_{+} = (X_{\underline{A}}, R_{\underline{\underline{A}}}, A_{+})$ where:

1. $X_{\underline{A}}$ is the Stone space of $\underline{A}$ as given in Definition 2.33,
2. A_{+} is the set of clopen sets, namely $\{\|a\| \mid a \in A\}$, and
3. for $i \in Idx$, $R_i = \{\langle x, y\rangle \in {}^{2}X_{\underline{A}} \mid (\forall a \in A)\,[I_i(a) \in x \Longrightarrow a \in y]\}$.

Full frames were introduced in this thesis purely as an attendant to the discussion of the frame/algebra duality since we could not get the functor $^{+}$ to undo $_{+}$ with purely full frames. While the duality holds between finite full frames and finite algebras, it fails between infinite full frames and infinite algebras. For consider an infinite algebra $\underline{A}$. It will have size $\mathrm{card}(A)$ and so the space of ultrafilters of $\underline{A}$ will be, in most circumstances, of size $2^{\mathrm{card}(A)}$. So moving to the full frame has increased the underlying cardinality exponentially. Moving back, via the $^+$ functor will take us to an underlying boolean algebra $\mathcal{P}(\mathrm{ult}(\underline{A}))$ of size $2^{2^{\mathrm{card}(A)}}$. Given this type of growth we cannot hope to get back to where we started to complete the duality circle.

We do have a working duality for certain types of first order frames however (see, e.g., GOLDBLATT's [32, Chapter 1]):

Theorem 3.33. *Suppose that $\underline{\underline{A}}$ is a normal $\mathcal{S}$-algebra.*[8] *Then $\underline{\underline{A}}_+$ is a descriptive first order frame and $(\underline{\underline{A}}_+)^+ \cong \underline{\underline{A}}$.*

Conversely if F is a descriptive first order frame then $(F^+)_+ \cong F$.

Sometimes, however, we do want to consider the duality between full frames and the underlying algebras to the extent that we identify full frames with their dual algebras.

Definition 3.34. Suppose that $\underline{\underline{A}} = (\underline{A}, \underline{I})$ is such that $\underline{A} \cong \underline{\mathcal{P}}(X)$. Then we can consider $\underline{\underline{A}}_r = (X, \underline{R})$ where for $i \in \mathit{Idx}$,

$$R_i = \{\langle x, y\rangle \mid (\forall a \in A)\,[x \in I_i(a) \Longrightarrow y \in a]\}\,.$$

In the event that $(\underline{\underline{A}}_r)^+ \cong \underline{\underline{A}}$ we say that $\underline{\underline{A}}$ is a *complex algebra* and that $\underline{\underline{A}}$ and F are *essentially the same*.

When a frame and an algebra are essentially the same the algebra can readily be constructed from the frame and vice versa, so there would be no harm in using them interchangeably. Note that every full frame F and its dual algebra F^+ are always essentially the same.

We know that we can relate $\mathcal{S}$-algebras with homomorphisms and that there are various operations we can perform in the variety of $\mathcal{S}$-algebras. Examples of such operations include that of finding all the homomorphic images of an $\mathcal{S}$-algebra, finding the product of a set of $\mathcal{S}$-algebras, and finding an $\mathcal{S}$-algebra's collection of $\mathcal{S}$-subalgebras. Through the duality given above, these have natural relational frame semantics analogs which we list here.

Firstly though, let us show how the operations of $^+$ and $_+$ are contravariant functors between the categories of $\mathcal{S}$-algebras and relational (first order) frames. We have already seen that $^+$ and $_+$ are mappings, so now we need to see how they map the morphisms of these categories. We need a definition of morphism in order to do this.

Definition 3.35. A map $f : (X, \underline{R}, \mathbb{P}) \longrightarrow (Y, \underline{S}, \mathbb{Q})$ is a *frame homomorphism in index $i \in \mathit{Idx}$* iff it satisfies the following:

[8] *An $\mathcal{S}$-algebra $\underline{\underline{A}}$ is normal iff $\mathcal{L}(\underline{\underline{A}})$ is a normal logic, that is, the algebra satisfies the laws equivalent to the normality axioms.*

1. $(\forall x, y \in X)\,[\langle x, y\rangle \in R_i \Longrightarrow \langle f(x), f(y)\rangle \in S_i]$,
2. $(\forall x \in X, y \in Y)\,[\langle f(x), y\rangle \in S_i \Longrightarrow$
   $(\exists z \in X)\,[f(z) = y \text{ and } \langle x, z\rangle \in R_i]]$, and
3. $(\forall Z \in \mathbb{Q})\,[f^{-1}[Z] \in \mathbb{P}]$.

In the case of full frames we omit the last condition, so a map $f : (X, \underline{R}) \longrightarrow (Y, \underline{S})$ is a *frame homomorphism in index* i iff the map satisfies 1 and 2 above.

We say that a map $f : (X, \underline{R}, \mathbb{P}) \longrightarrow (Y, \underline{S}, \mathbb{Q})$ is a *frame homomorphism* iff it is a frame homomorphism in all indices $i \in Idx$. Similarly for a map $f : (X, \underline{R}) \longrightarrow (Y, \underline{S})$.

The last condition does tell us how to turn a frame homomorphism into a map between the dual algebras:

Definition 3.36. Suppose that $f : F \longrightarrow G$ is a map between first order (or full frames) then $f^+ : G^+ \longrightarrow F^+$ is defined by

$$f^+(Z) = f^{-1}[Z].$$

These frame homomorphisms are more commonly called *p-morphisms* (see e.g., CHELLAS's [13]) or *bounded homomorphisms* (see e.g., GOLDBLATT [32]).

Recall from Definition 2.55 that every map $f : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ naturally gives rise to a map f_+ between the Stone spaces of $\underline{B}$ and $\underline{A}$ respectively.

Now the following theorems can easily be proven, see e.g., GOLDBLATT's book [32].

Theorem 3.37. *Suppose that* $f : \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ *is an* $\mathcal{S}$*-algebra homomorphism, then* $f_+ : \underline{\underline{B}}_+ \longrightarrow \underline{\underline{A}}_+$ *is a frame homomorphism. Moreover,*

1. *if f is into then f_+ is onto,*
2. *if f is onto then f_+ is into, and*
3. *if f is an isomorphism, then f_+ is an isomorphism.*

Suppose that $f : F \longrightarrow G$ is a frame homomorphism, then $f^+ : G^+ \longrightarrow F^+$ is an $\mathcal{S}$*-algebra homomorphism. Moreover,*

1. *if f is into then f^+ is onto,*
2. *if f is onto then f^+ is into, and*
3. *if f is an isomorphism, then f^+ is an isomorphism.*

This duality naturally tells us how the logics of frames relate to each other: Suppose that G is a frame homomorphic image of F via some frame homomorphism f. Thus $f^{+} : G^{+} \longrightarrow F^{+}$ is an into $\mathcal{S}$-algebra homomorphism, which is to say that G^{+} really is an $\mathcal{S}$-subalgebra of F^{+}. This naturally means that $\mathcal{L}(F^{+}) \subseteq \mathcal{L}(G^{+})$. Since our initial definition of satisfaction and truth in frames depends on truth in their dual algebras, we immediately get the following proposition.

Proposition 3.38. *Suppose that the full [first order] frame G is a frame homomorphic image of the full [first order] frame F. Then $\mathcal{L}(F) \subseteq \mathcal{L}(G)$.*

Another operation on algebras that has an analog in frames is that of the algebraic product:

Definition 3.39. Suppose that $\langle F_k = (X_k, \underline{R}_k, \mathbb{P}_k) \mid k \in \mathcal{K}\rangle$ is a sequence of first order frames. Then we define

$$\biguplus_{i \in \mathcal{K}} F_k,$$

the *disjoint union* of this sequence, to be $F = (X, \underline{R}, \mathbb{P})$ where

1. $X := \bigcup_{k \in \mathcal{K}} (X_k \times \{k\})$,
2. for $i \in Idx$, $R_i = \{\langle\langle x,k\rangle, \langle y,k\rangle\rangle \mid \langle x,y\rangle \in R_{k_i}, k \in \mathcal{K}\}$, and
3. $\mathbb{P} = \{Z \subseteq X \mid (\forall k \in \mathcal{K})\,[\{x \in X_k \mid \langle x,k\rangle \in Z\} \in \mathbb{P}_k]\}$.

For F a full frame we omit the last criterion since it is naturally satisfied.

Then we have the following result. (See e.g., KRACHT's [50, p. 171] for a very category-theoretic presentation.)

Theorem 3.40. *For $\langle F_k \mid k \in \mathcal{K}\rangle$ a sequence of frames, the following equality holds:*

$$\left(\biguplus_{k \in \mathcal{K}} F_k\right)^{+} = \prod_{k \in \mathcal{K}} {F_k}^{+}.$$

Again, because satisfaction of formulae is based on satisfaction in the associated algebra, we get the following result.

Proposition 3.41. *For $\langle F_k \mid k \in \mathcal{K}\rangle$ a sequence of frames,*

$$\mathcal{L}\left(\biguplus_{k \in \mathcal{K}} F_k\right) = \bigcap_{k \in \mathcal{K}} \mathcal{L}(F_k).$$

The last operation is really just a specialised type of homomorphic image:

Definition 3.42. A first order frame $G = (Y, \underline{S}, \mathbb{Q})$ is a *generated subframe* of $F = (X, \underline{R}, \mathbb{P})$ iff G is a subframe of F and the map id : $Y \longrightarrow X$ is a frame homomorphism. We say that G is *generated by the points* in $W \subseteq X$ iff $Y = \bigcup_{\alpha<\omega} W_\alpha$, where

$$W_0 = W, \text{ and}$$

$$W_{\alpha+1} = \bigcup_{i\in Idx, w\in W_\alpha} R_i(w).$$

As a definition this is not that revealing and we provide the following equivalent which tells us that a generated subframe is a subframe which is "closed upwards" or that we can start at some collection W and "generate" G by following the various successors of the elements of W.

Proposition 3.43. *A frame $G = (Y, \underline{S}, Q)$ is a generated subframe of $F = (X, \underline{R}, P)$ iff*

1. $Y \subseteq X$,
2. *for all* $i \in Idx$, $S_i = R_i \cap (Y \times Y)$, *and*
3. *for all* $i \in Idx$, $(\forall y \in Y, x \in X)[\langle y, x\rangle \in R_i \Longrightarrow x \in Y]$.

Since a generated subframe G of F really has an associated one-one frame homomorphism going from G to F, we get that G^+ is a homomorphic image of F^+. Thus:

Proposition 3.44. *If G is a generated subframe of F then $\mathcal{L}(F) \subseteq \mathcal{L}(G)$.*

We have one last definition which, while essentially implicit in the earlier work, should be spelled out as the whole of Chapter 9 is devoted to it.

Definition 3.45. A map $f : F \longrightarrow G$ between frames is called an *isomorphism* iff f is a one-one onto frame homomorphism. It is called an *automorphism* iff $F = G$.

3.7 Cardinality Questions and Set Theory

The canonical frames we discussed earlier are well known to most practitioners of modal logic. This is mostly because their ease of use and universality facilitates completeness proofs. Certainly the underlying construction is conceptually straightforward, however it is a mathematically deep procedure, not in the least because the construction is non-effective and is closely linked to the mathematically mysterious powerset operation. Further evidence for this can be found in the number of open problems regarding the general structure of Canonical Frames.

An example is Conjecture 3.28 "If a modal logic is canonical then it is elementary" and we saw GOLDBLATT's Theorem 3.29 that if it is elementary then it is described by the elementary class of its canonical frame. Even though this almost compels us to believe Conjecture 3.28, it gives us little understanding of the canonical frame's true nature beyond the knowledge that it is "saturated" in the sense of KIT FINE's proof, in [22], of elementarity implying canonicity.

Our understanding of canonical frames is further complicated by the intrusion of cardinality considerations. For instance, another outstanding problem in the study of canonical frames is the question of whether the cardinality of the underlying language affects the logic verified by the canonical frame. More precisely:

Conjecture 3.46. If L, a normal modal logic, is λ-canonical for some infinite cardinal λ then it is κ-canonical for all cardinals $\kappa > \lambda$.

In some sense this conjecture is a derivative of Conjecture 3.28 since if a logic is elementary then it would follow that it is canonical in all infinite cardinals, however, if canonicity implied elementarity, the proof of this fact would *probably* be based solely on canonicity with respect to one particular infinite set of propositional letters, and so based on canonicity in one infinite cardinal.

We could even go further and ask:

Question 3.47. For L, a normal modal logic, and κ, λ different cardinals, is F^L_κ elementary equivalent to F^L_λ?

It may well be that the answer to these questions will be set theoretic in nature or conceivably even independent of ZFC. After all, if the cardinality of the language is κ, then the cardinality of the canonical frame (underlying Stone space) is 2^κ suggesting that some property reminiscent of the Continuum Hypothesis may be at work.

Straying further down this path, we might wonder what happens in the presence of new axioms, i.e., statements independent of ZFC. For instance, we may wish to answer the very pruned down question of "Do F^L_ω and $F^L_{\omega_1}$ verify the same logic?" If we suspect that there is something 'independent' going on, we may wish to see what happens in the presence of the axiom '$2^\omega = 2^{\omega_1}$'. Perhaps we could even prove that F^L_ω is isomorphic to $F^L_{\omega_1}$.

In Chapter 9 we show that we can answer this question with a 'yes' for the trivial logics above and around S5 and 'no' for those logics below. Outside of this range a partial answer can be obtained by looking at *standard* versus *non-standard* isomorphisms.[9]

[9]Recall from Section 2.10 that a map g between Stone spaces (and hence between canonical

Even if our approach to these canonical frame problems avoids appeal to these extra-ZFC notions, we will still be strongly tempted to construct some kind of a map between F^L_ω and $F^L_{\omega_1}$, and we will naturally be inclined to require that such a map be a frame homomorphism or at least a map which is almost a frame homomorphism. Again, because of the cardinality difference we are unlikely to have much luck if we take these maps to be standard.

Question 3.48. Are there any non-standard frame homomorphisms between canonical frames, and if so what properties do they have?

This is also a difficult question which we cannot answer in anywhere near a satisfactory manner, however we do make a start, both by looking at non-standard non-frame-homomorphic maps in Chapter 8 and by looking at standard versus non-standard isomorphisms in Chapter 9.

This thesis details some interesting results on these questions and conjectures and it is the hope of the author that by studying simpler problems like Question 3.48 we will help bring about a solution to the elusive yet compelling Conjecture 3.28.

frames) is called standard iff it is induced by a map between the underlying algebras—i.e., iff it is of the form $g = f_+$.

Chapter 4

Neighborhood Semantics

4.1 Introduction

This chapter will introduce the notion of a neighborhood frame that will be used in subsequent chapters. As we shall see, the resultant semantics generalise the more common and well known relational semantics of the last chapter yet they are still somewhat removed from the most general algebraic semantics of Chapter 2.

One central notion which we will introduce here is the concept of neighborhood canonicity. This has a natural place in proving neighborhood completeness of various logics. As far as the author has been able to determine, the concept used here was first introduced by BRIAN F. CHELLAS in his book [13]. It is a little different to the notion introduced in the previous chapter as it allows some flexibility in what a canonical frame is, yet it does still force us to deal with the underlying Stone space. Clearly it is a useful concept as it generalises the notion of relational canonicity and so gives us yet another tool in which to investigate the phenomenon of canonicity.

The chapter will conclude with a discussion of the general relationship between neighborhood and relational frames, thus indicating how the descent from 'well understood' relational frames to neighborhood frames often introduces an element of mystery.

For this chapter let us assume that we have one fixed infinite set of propositional variables P. Thus all our results can be thought of as being prefixed by a universal quantification over all infinite P. Also, we will suppress the subscript $_P$ on various symbols where appropriate, so $\underline{A}^L$, $\underline{I}^L$, $\underline{B}^L$, $\underline{J}^L$, and X^L can all be thought to have an invisible subscript $_P$.

4.2 Algebraic Frames, Completeness and Canonicity

There is a duality between neighborhood frames and what are sometimes called full algebras. This duality, comprehensively investigated by KOSTA DOŠEN [17], is so strong that we will often forget about the associated neighborhood structure and just deal with their algebraic equivalents.

Recall Definition 2.21 which defined a power set boolean algebra to be a boolean algebra which is isomorphic to $(\mathcal{P}(X), \cap, \cup, \neg, X, \varnothing)$, where X is an arbitrary set, and $\mathcal{P}(X)$ is the power set of X.

Recall our remarks in Section 2.8 which arrived at the fact that all intensional logics were verified by their Lindenbaum algebras and so were complete with respect to their algebraic semantics. While this worked easily, we did not introduce a concept of strong completeness—something which does not make sense unless we have something like a frame underlying it all (c.f. Definition 3.9). The reason is as follows:

We could say that L is weakly [strongly] complete iff for every L-consistent formula φ [set Σ], there is a valuation v on an $\mathcal{S}$-algebra $(\underline{A}, \underline{I})$, $(\underline{A}, \underline{I}) \models L$, such that $\overline{v}(\varphi) \neq 0_A$ [$\overline{v}(\varphi) \neq 0_A$ for all $\varphi \in \Sigma$]. Then the weak and strong completeness of every intensional logic L would be immediate as $(\underline{A}_P^L, \underline{I}_P^L)$ and v_P^L would witness this fact for a suitably large P.

As will be seen in the next section, it is more interesting to define completeness in terms of algebraic frames; note how the definitions given here are in keeping with those in 2.46 and how, qualitatively, these definitions are similar to those in Chapter 3.

Definition 4.1. An *algebraic frame* is an $\mathcal{S}$-algebra $(\underline{A}, \underline{I})$ with the property that $\underline{A}$ is a power set boolean algebra.

An algebra that is an algebraic frame is sometimes called a *full* algebra.

Here, if $A = \mathcal{P}(X)$ then each element of X can be considered to be 'a point in the frame'.

Definition 4.2. An intensional logic L is *weakly neighborhood complete* iff for every L-consistent formula $\varphi \in \mathcal{S}(P)$, there is an algebraic frame $(\underline{A}, \underline{I})$, $(\underline{A}, \underline{I}) \models L$, $\underline{A} = \underline{\mathcal{P}}(X)$, a valuation v, and an $x \in X$ such that $x \in v(\varphi)$.

Definition 4.3. An intensional logic L is *strongly neighborhood complete* iff for every L-consistent set $\Sigma \subseteq \mathcal{S}(P)$, there is a frame $(\underline{A}, \underline{I})$, $(\underline{A}, \underline{I}) \models L$, $\underline{A} = \underline{\mathcal{P}}(X)$, a valuation v, and an $x \in X$ such that $x \in v(\varphi)$ for each $\varphi \in \Sigma$.

It is no longer easy to show that L is complete in either sense.

Just as with the relational case we have a notion of the finite frame property, however it is not so different from what we had for algebraic structures:

Definition 4.4. A logic L is said to have the *finite neighborhood frame property* iff L is complete with respect to its class of finite algebraic frames. That is, if $\varphi \notin L$, then there is a finite algebraic frame $(\underline{A}, \underline{I})$ such that $(\underline{A}, \underline{I}) \models L$ and $(\underline{A}, \underline{I}) \not\models \varphi$.

This is not really a new concept because every finite algebra is also a finite frame. Hence again, the finite frame property corresponds to the finite model property. We do have the following result though:

Proposition 4.5. *If an intensional logic L has the finite model property (or finite frame property) then it is also neighborhood weakly complete.*

Proof. Suppose that L has the finite frame property. That L is neighborhood weakly complete follows with little work, for if ψ is an L-consistent formula, then $\varphi = \neg\psi$ is a non-theorem of L. Applying the finite frame property to φ we have a finite frame $(\underline{B}, \underline{J})$ with $(\underline{B}, \underline{J}, v) \not\models \varphi$ for some v. Thus $v(\varphi) \neq \top_B$ and so if $\underline{B}$ really is the power set $\mathcal{P}(X)$ say, then there is some $x \in X$ with $x \in v(\neg\varphi) = v(\psi)$, as required. □

We can now move on to the notions of neighborhood canonicity.

Definition 4.6. For L an intensional logic, we define $\underline{B}^L = \underline{\mathcal{P}}(X^L)$.

Now, $\underline{B}^L$ is a good candidate for showing strong completeness over all L-consistent sets Σ, but $\underline{B}^L$ must first be outfitted with a $\underline{J}^L$ agreeable with $\underline{I}^L$.

Definition 4.7. An intensional logic L is *κ-neighborhood-canonical* iff for card$(P) = \kappa$ there is a $\underline{J}^L$ so that $(\underline{A}^L, \underline{I}^L)$ is an $\mathcal{S}$-subalgebra of $(\underline{B}^L, \underline{J}^L)$ and $(\underline{B}^L, \underline{J}^L) \models L$. We say that L is *canonical* iff it is canonical in all cardinals κ.

This definition may seem slightly strange at first, however, the standard definition of canonicity for neighborhood semantics will be examined in the next section and readily seen to be equivalent to this one.

We included the term 'neighborhood' in the definitions here to distinguish this type of completeness from 'relational' completeness—the use of the term 'neighborhood' itself will be justified in the next section. If the context, such as when L has not been constrained to be normal, makes it clear which type of completeness we are talking about, we omit the reference 'neighborhood'.

We note the immediate result:

Proposition 4.8. *If an intensional logic is neighborhood canonical then it is strongly neighborhood complete.*

As with the relational semantics, there is a relevant concept which will come up occasionally and that is the concept of a logic being neighborhood complex.

Definition 4.9. A logic L is *neighborhood complex* iff every algebra $(\underline{A}, \underline{I})$ for L can be homomorphically embedded into an algebraic frame $(\underline{B}, \underline{J})$ for L.

Certainly this is a more general concept than that of canonicity but which turns out to be a concept of interest for normal modal logics—we will note in chapters 5 and 6 that what are known as non-iterative logics, and what we will call even logics, have this property. But for the moment, we will content ourselves with explicitly noting:

Proposition 4.10. *If L is neighborhood canonical then it is complex.*

4.3 Neighborhood Frames

In proving most of the main results of this thesis we will avoid the direct use of neighborhood frames. However, it is important to see how the algebraic constructions introduced here are equivalent to the neighborhood frames approach and we will use neighborhood frames in our informal discussions as they are more intuitive. Moreover, the neighborhood frames approach does underline the need for a strong completeness result and this approach will give the desired form for our result.

It should be stressed again that we are only considering the duality between frames (powerset, or full algebras) and full neighborhood frames. Of course, this duality can be made precise for all types of neighborhood frames and all algebras; the reader is advised to consult DOŠEN [17].

Definition 4.11. A *neighborhood frame* for S is a structure of the form $(X, \underline{N})$, where $\underline{N} = \langle N_i \rangle_{i \in Idx}$ and each $N_i : X \longrightarrow \mathcal{P}(\mathcal{P}(X)^{n_i})$.

Definition 4.12. A *valuation* v on $(X, \underline{N})$, a neighborhood frame, is a map that assigns $v(p) \subseteq X$ to each propositional letter p. The valuation v is extended to $v : \mathcal{S}(P) \longrightarrow \mathcal{P}(X)$ as follows:

1. $v(\psi \wedge \varphi) = v(\psi) \wedge v(\psi)$, etc., and
2. for $i \in Idx$, $v(\Box_i(\overline{\psi})) = \{x \mid v(\overline{\psi}) \in N_i(x)\}$.

These neighborhood models are called *minimal models* by CHELLAS in [13] and the semantics are sometimes called Scott-Montague semantics.

Note that by Definition 2.46 this gives us notions of satisfaction, completeness, and even allows us to talk about $\mathcal{L}(F)$ for F a neighborhood frame.

Without loss of generality, in what follows take $(\underline{A}, \underline{I})$ to be an algebraic frame and take A to actually be $\mathcal{P}(X)$ for some set X.

Each frame $(\underline{A}, \underline{I})$ naturally defines a neighborhood frame and vice versa:

Definition 4.13. Given an algebraic frame $(\underline{A}, \underline{I})$, define $(\underline{A}, \underline{I})_+$, the neighborhood frame derived from $(\underline{A}, \underline{I})$, as follows: Let $(\underline{A}, \underline{I})_+ = (X, \underline{N})$ where, for $i \in \mathit{Idx}$,

$$N_i(x) = \{\overline{a} \mid a_1, a_2, \ldots, a_{n_i} \subseteq X, x \in I_i(\overline{a})\}.$$

If v is a valuation on $(X, \underline{N}) = (\underline{A}, \underline{I})_+$ it is also a valuation on $(\underline{A}, \underline{I})$. By noting that, for $i \in \mathit{Idx}$,

$$\begin{aligned} v(\Box_i(\overline{\psi})) &= \{x \mid v(\overline{\psi}) \in N_i(x)\} \\ &= \{x \mid v(\overline{\psi}) \in \{\overline{a} \mid x \in I_i(\overline{a})\}\} \\ &= \{x \mid x \in I_i(v(\overline{\psi}))\}, \end{aligned}$$

we see that the extensions of v to $(\underline{A}, \underline{I})$ and $(\underline{A}, \underline{I})_+$ are identical and so precisely the same formulae hold over $(\underline{A}, \underline{I})_+$ as over $(\underline{A}, \underline{I})$.

Definition 4.14. If $(X, \underline{N})$ is a neighborhood frame then set $(X, \underline{N})^+ = (\underline{A}, \underline{I})$ where $\underline{A} = \underline{\mathcal{P}}(X)$ and $I_i(\overline{a}) = \{x \mid \overline{a} \in N_i(x)\}$.

In a similar way, we can see that the same formulae hold over $(X, \underline{N})$ and $(X, \underline{N})^+$. It is also not hard to see that ${(\underline{A}, \underline{I})_+}^+ = (\underline{A}, \underline{I})$ and ${(X, \underline{N})^+}_+ = (X, \underline{N})$.

A central quest of this thesis is the quest to find out which logics L are canonical. In the sense of neighborhood semantics, this means that we can find an $\underline{N}^L$ such that $(X^L, \underline{N}^L)$ satisfies:

1. $(X^L, \underline{N}^L) \models L$, and
2. $\|\overline{\psi}\| \in N_i^L(x) \iff x \in \|\Box_k(\overline{\psi})\|$ for each $i \in \mathit{Idx}$.

This is the definition of canonicity that can be found in CHELLAS [13, p. 252], modified to take into account multi-adic, multi-modal operators.

In the light of what we have just seen, it will be enough to find a $\underline{J}^L$ so that $(\underline{B}^L, \underline{J}^L) \models L$ and $(\underline{A}^L, \underline{I}^L)$ is an $\mathcal{S}$-subalgebra of $(\underline{B}^L, \underline{J}^L)$, for then setting $(X^L, \underline{N}^L) = (\underline{B}^L, \underline{J}^L)_+$ will give us $(X^L, \underline{N}^L) \models L$ and

$$\begin{aligned} \|\overline{\psi}\| \in N_i^L(x) &\iff x \in J_i(\|\overline{\psi}\|) \\ &\iff x \in I_i(\|\overline{\psi}\|) \\ &\iff x \in \|\Box_i(\overline{\psi})\|. \end{aligned}$$

This is precisely the requirement that L is canonical.

Since we have this close duality between neighborhood semantics and algebraic frame semantics we will switch between the two depending on our needs. Typically it will be most convenient to use neighborhood frames when discussing the import of certain results and, conversely, we will most likely use algebraic frame semantics in our technical derivations where the consideration of 'neighborhoods' removes us a little too much from the mathematical undercurrent.

4.4 Elementary Canonicity and Completeness Results

Part III of CHELLAS's book [13] is devoted to an elementary introduction to the study of neighborhood semantics or, as he calls them, minimal models. We will reiterate some of the canonicity results noted there and then go on to provide some other elementary comments on canonicity and completeness that have been made.

Firstly, note that unlike relational semantics, each formula in $\mathcal{S}(\omega)$ naturally interprets as a condition on an algebraic frame and therefore a neighborhood frame. In relational semantics we had to search around (and sometimes even failed) to find a first order condition on a frame that guarantees that the frame satisfied our logic, whereas with the algebraic semantics, and hence the algebraic frame semantics, the formula itself is the condition on the algebra that guarantees that the frame satisfies the formula. A simple translation will then give us the condition on the neighborhood frame, expressed in terms of the neighborhood function, that guarantees that the frame will satisfy the formula.

This relationship is made precise in ROY A. BENTON's paper [6], however, the translation is natural, and we will content ourselves with just listing, in Table 4.1, the translations for a few important formulae for the mono-modal case (we assume that each of the algebraic and neighborhood translations are universally quantified).

CHELLAS notes in [13] that logics like EN are almost trivially canonical. To do this he uses what can arguably be called the smallest canonical frame.[1] Over the canonical Stone space X^L he sets

$$N^L(x) = \{||\varphi|| \mid \Box\varphi \in x\}$$

and so if the structure (X^L, N^L)[2] really is a frame for L then it is clearly a

[1] Of course, CHELLAS does not have proprietary rights over this technique and the one given below since they appear in SEGERBERG's [72].

[2] This structure needs to be proven to be canonical, however before that point it is clearly a candidate for a canonical frame and so we could use the apt term *candidate L-canonical* frame, which was coined by BENTON in [6]

Axiom	Formula	Algebraic Translation	Neighborhood Frame Translation
M	$\Box(p \wedge q) \to (\Box p \wedge \Box q)$	$I(a \wedge b) \leqslant I(a) \wedge I(b)$	$Y \cap Z \in N(x) \Longrightarrow Y \in N(x)$ and $Z \in N(x)$
C	$(\Box p \wedge \Box q) \to \Box(p \wedge q)$	$I(a) \wedge I(b) \leqslant I(a \wedge b)$	$Y \in N(x)$ and $Z \in N(x) \Longrightarrow Y \cap Z \in N(x)$
N	$\Box\top$	$I(\top) = \top$	$X \in N(x)$
K	$\Box(p \to q) \to (\Box p \to \Box q)$	$I(\neg a \vee b) \wedge I(a) \leqslant I(b)$	$(X - Z) \cup Y \in N(x)$ and $Z \in N(x) \Longrightarrow Y \in N(x)$
4	$\Box p \to \Box\Box p$	$I(a) \leqslant I(I(a))$	$Y \in N(x) \Longrightarrow \{y \in X \mid Y \in N(y)\} \in N(x)$
D	$\Box p \to \Diamond p$	$I(a) \leqslant \neg I(\neg a)$	$Y \in N(x) \Longrightarrow (X - Y) \notin N(x)$

Table 4.1: Algebraic and neighborhood translations of common axioms. (Note: X is the underlying set in the neighborhood frame)

canonical frame for L. In the case of simple logics it clearly is a canonical frame.

Proposition 4.15. *If L is one of the logics* EN, EC, *or* ECN, *then the frame (X^L, N^L) is a frame for L and so the logics are canonical.*

Proof. We prove the result for $L =$ ECN however we suppress the superscript ^{L} on X and N. We must verify the two conditions corresponding to C and N.

$(\forall x \in X)\,[X \in N(x)]$.

Let $x \in X$. Since $\Box\top \in L$ we have that $\Box\top \in x$ and so $X = ||\top|| \in N(x)$.

$(\forall x \in X)\,(\forall Y, Z \subseteq X)\,[Y \in N(x) \text{ and } Z \in N(x) \Longrightarrow Y \cap Z \in N(x)]$.

Let $x \in X$, let $Y, Z \subseteq X$, and suppose that $Y \in N(x)$ and $Z \in N(x)$. Thus there are $\varphi, \psi \in \mathcal{S}(P)$ such that $Y = ||\varphi||$ and $Z = ||\psi||$, for some $\Box\varphi \in x$ and $\Box\psi \in x$. But by C we can then conclude that $\Box(\varphi \wedge \psi) \in x$ so $Y \cap Z = ||\varphi \wedge \psi|| \in N(x)$.

$\Box$

One can then quickly see that simple logics which only have axioms of the form

$$\Box p_0 \wedge \cdots \wedge \Box p_{n-1} \to \varphi$$

(where the propositional letters of φ are among $\{p_0, \ldots, p_{n-1}\}$) can be quickly shown to be canonical through use of this *smallest canonical frame*. Problems start when we have the mixing of propositional letters in the antecedents. A simple example of this is the logic EM, but this logic can be handled by adding to the elements of N^L in a naïve way and this forms a new (X^L, N^L) which is what CHELLAS calls the *supplementation* of the minimal canonical frame:

$$N^L(x) = \left\{Y \subseteq X^L \mid (\exists\varphi \in \mathcal{S}(P))\,[\Box\varphi \in x \text{ and } ||\varphi|| \subseteq Y]\right\}.$$

That is we take the smallest model and, as CHELLAS would say, supplement it by closing $N^L(x)$ under supersets.

Definition 4.16. A neighborhood frame (X, N) is called *supplemented* iff

$$(\forall x \in X)\,(\forall Y \in N(x))\,(\forall Z \in \mathcal{P}(X))\,[Y \subseteq Z \Longrightarrow Z \in N(x)].$$

Clearly our frame (X^L, N^L) is supplemented.

Proposition 4.17. *For $L =$ EC and (X^L, N^L) the neighborhood frame given above, we have that $(X^L, N^L) \models L$, i.e., (X^L, N^L) is a canonical frame for L.*

Proof. Again we suppress the superscript ^{L}. We need only verify the condition for $\Box(p \wedge q) \to (\Box p \wedge \Box q)$:
$(\forall x \in X;\ Y, Z \subseteq X)\,[Y \cap Z \in N(x) \Longrightarrow Y \in N(x)$ and $Z \in N(x)]$

Let $x \in X$, let $Y, Z \subseteq X$, and suppose that $Y \cap Z \in N(x)$. Thus there is a $\varphi \in \mathcal{S}(P)$ such that $\|\varphi\| = Y \cap Z$ and $\Box\varphi \in x$. But both $Y, Z \supseteq Y \cap Z = \|\varphi\|$ so we then have that $Y, Z \in N(x)$ as desired.

□

Of course things get more complicated as we add axioms, for instance if we were to look at a canonical frame for **EMC** we would need, what CHELLAS calls a *quasi-filtering* of our "smallest canonical frame." This is where we take the smallest canonical frame and close each $N^L(x)$ under finite intersections, and then perform a supplementation—close under supersets.

A final type of simple canonical frame construction that we will mention here is that which is provided in BENTON's paper [6], where he takes all non-effable sets (not of the form $\|\varphi\|$ for some φ) and puts them in $N^L(x)$ regardless. This will quite clearly be the *largest candidate L-canonical frame*. BENTON shows, along the lines given above, that if L includes formulae of the form

$$\varphi \to \Box p_0 \vee \cdots \vee \Box p_{n-1}$$

(where the propositional variables of φ are all in $\{p_0, \ldots, p_{n-1}\}$) then the largest candidate L-canonical frame verifies the condition corresponding to this formula. Again, what we see here is a kind of lack of mixing of propositional variables.

The canonical constructions we have seen so far are still rather simplistic as we make one global change to the canonical frame and then hope that this creates a candidate L-canonical frame which satisfies L. Unfortunately this is too much to hope for (even in some simple logics). In the same paper [6] BENTON shows that **EK** cannot be treated in this manner and goes on to carry out a very delicate construction of a canonical frame for **EK** from the smallest candidate L-canonical frame.

The logic **EK** is an example of a class of logics known as *non-iterative*. This is a class where intensional operators are not allowed in the scope of intensional operators within some axiomatisation for each logic in the class. These are reasonably simple logics and so it is not surprising that they are all complete with respect to the neighborhood frame semantics—this was shown by DAVID LEWIS in [58] where he took the canonical *model* for a non-iterative logic and filtered it down to a finite frame for that logic. We will revisit that technique in Chapter 5 where we will use the same overall idea to show that this implies canonicity and the complexity of our proof will be testament to how the neighborhood canonicity question becomes complicated.

We have seen that canonicity, and hence completeness, for some logics is reasonably straightforward. In fact, it has been almost immediate and so some people may think that this shows that there is little in neighborhood semantics beyond that given by algebraic semantics.[3] This would be wrong, as BENTON's complicated and delicate proof that EK is canonical (and so strongly complete) reveals. Moreover, we will see in the next section that not all logics are complete with respect to the neighborhood frame semantics.

4.5 Relational Frames and Neighborhood Frames

As we originally stated at the start of this chapter, a neighborhood frame is a generalisation of a relational frame and here we will make this precise. Of course, when talking about relational frames we move into the realm of normal multi-modal logics—and this means that we take all our modal operators to be unary, and our logics to be normal.

Definition 4.18. Let $(X, \underline{R})$ be a relational frame. Then the *neighborhood frame associated with* $(X, \underline{R})$ is the neighborhood structure $(X, \underline{N})$ where, for $i \in Idx$,

$$N_i(x) = \{Y \subseteq X \mid R_i(x) \subseteq Y\}.$$

If this happens, we say that $(X, \underline{N})$ and $(X, \underline{R})$ are equivalent.

We then get the following immediate result which shows that as far as the language is concerned there is no difference between a relational frame and its associated neighborhood frame:

Proposition 4.19. *Let $(X, \underline{R})$ be a relational frame and $(X, \underline{N})$ its associated neighborhood frame. If v is a valuation on one of the frames $(X, \underline{R})$ and $(X, \underline{N})$ then it is also a valuation on the other frame and*

$$(\forall \varphi \in \mathcal{S}(P))\,(\forall x \in X)\,[(X, \underline{R}, v) \vDash_x \varphi \iff (X, \underline{N}, v) \vDash_x \varphi].$$

We can reverse the process given above for if $(X, \underline{N})$ is the neighborhood frame associated with $(X, \underline{R})$, then for each $x \in X$, $R(x) = \bigcap N(x)$. However we cannot adopt such an approach with an arbitrary neighborhood frame since a full equivalence would require that each $N(x)$ is closed under arbitrary intersections and under supersets:

[3] Note that I am not saying "algebraic frame semantics" here.

Definition 4.20. A neighborhood frame $(X, \underline{N})$ is *augmented*[4] iff is it is supplemented and each $N_i(x)$ is closed under arbitrary intersections.

Proposition 4.21. *Each augmented neighborhood frame is equivalent to a relational frame and vice versa.*

Of course, we could have a perfectly good neighborhood frame for a normal multi-modal logic where some neighborhoods are not closed under arbitrary intersections, and we will see examples of this later. One thing we should note is:

Proposition 4.22. *Let $(X, \underline{N})$ be a neighborhood frame. If $\mathcal{L}((X, \underline{N}))$ is normal then $(X, \underline{N})$ is supplemented.*

Thus we can test a neighborhood frame for a normal logic to see whether it is essentially relational just by looking to see if it is closed under arbitrary intersections.

One simple consequence of this fact is that any finite neighborhood frame for a normal modal logic is closed under arbitrary intersections and so it is always associated with a relational frame.

We will see in Section 5.5 that for some ordinary normal modal logics we can find a neighborhood frame for that logic which is so 'non-relational' that if we were to take its augmentation—closing its neighborhoods under arbitrary intersections and supersets—we would end up with almost trivial neighborhoods as each neighborhood would be very close to $\mathcal{P}(X)$.

A more involved and revealing illustration of the non-relational nature of most neighborhood frames can be found in the papers of MARTIN GERSON. In these papers, and also in his Ph.D. Thesis [26], he investigates the relationship between neighborhood semantics and other types of semantics for normal modal logics (including the algebraic and relational mentioned here). There are two major lessons from this work:

1. The neighborhood frame semantics is strictly stronger than the algebraic semantics. In [28] he looks at two logics, the Thomason logic (presented by S.K. THOMASON in [97]) and the Fine logic (presented by KIT FINE in [19]). Both are examples of logics which are incomplete with respect to the relational semantics. GERSON shows that they are both also incomplete with respect to the neighborhood semantics.

   We saw that via the Lindenbaum construction each normal modal logic is complete with respect to its algebraic semantics. GERSON was able to conclude that neighborhood frames are essentially richer structures than algebras.

[4] See CHELLAS [13, p. 220].

2. The neighborhood frame semantics is strictly weaker than the relational frame semantics. Gerson showed that there are normal modal logics (one above T [29] and one above S4 [27]) which are complete for the neighborhood frame semantics but incomplete for relational frames. He accomplished this by constructing neighborhood frames whose logics are naturally complete with respect to their neighborhood frame semantics and these logics are then shown, after a bit of work, to be incomplete with respect to the relational semantics. Similar work was carried out by DOV M. GABBAY who also provided [24] such a neighborhood frame and associated logic.

   This also establishes that there are sensible neighborhood frames whose neighborhoods are not always closed under arbitrary intersections—otherwise these frames could be turned into relational frames without affecting the logic.

The kind of incompleteness result presented by GERSON has been taken further by LILIA CHAGROVA in [12] where she has taken a result of W.J. BLOK [7] (that the *degree of relational frame incompleteness* of a normal modal logic L[5]—

$$\operatorname{card}\left(\{L' \mid (\forall F \text{ a relational frame})\ [F \models L \iff F \models L']\}\right)$$

—is either 1 or 2^ω) and shown that the analogous result holds for neighborhood frames. In fact, she showed that the degree of neighborhood frame incompleteness of a normal modal logic is equal to the degree of relational frame incompleteness of that logic.

The coercion process of turning a relational frame into a neighborhood frame also shows the following:

Proposition 4.23. *If a normal modal logic is relational canonical (in cardinal κ) then it is also neighborhood canonical (in cardinal κ).*

The converse is, however is not so straightforward.

Conjecture 4.24. If a normal modal logic is neighborhood canonical then it is also relational canonical.

If we add up the anecdotal evidence available to us there is little indication that this result does hold. After all, we have seen that relational frame semantics is stronger than neighborhood frame semantics so it does not seem unreasonable to expect that there is a logic which does have a canonical neighborhood frame yet it is not verified by its canonical relational frame. Remember

[5]With one unary modal operator.

that we have a lot of freedom when it comes to choosing a canonical neighborhood frame that might be suitable, however we have absolute constraints over which canonical relational frame we can choose. Why should they be related? Nevertheless, this is a natural question to ask and, in Chapter 6, we will show that the logic KMcK is a counter example to this conjecture.

One of the problems we encounter when making the jump from relational frame to neighborhood frame semantics is that we lose a number of familiar notions. For instance, we can no longer talk about elementary classes of neighborhood frames as there is no accessibility relation on which to hang our first order satisfaction. All we have are the conditions on sets in neighborhoods and these are essentially second order. Also missing is the notion of a generated subframe; as we will see in our appendix on the logic EK4 (Appendix A) every point in the frame could be important to the truth of some formulae and so no points can be dropped. We do have a notion analogous to a frame homomorphism (see DOŠEN [17]) but this notion is little more than the homomorphism between the associated algebraic frames, so again we have lost the geometric aspect.

We do, however have disjoint unions:

Definition 4.25. Suppose that $(X^k, \underline{N}^k)$, $k \in \mathcal{K}$ is a collection of neighborhood frames. Then we define their *disjoint union* to be

$$\biguplus_{k\in\mathcal{K}} (X^k, \underline{N}^k) = (X, \underline{N})\text{, where}$$

$$X = \bigcup_{k\in\mathcal{K}} X^k \times \{k\} \text{ and, for } i \in Idx,$$

$$N_i(\langle x, k\rangle) = \left\{Y \subseteq X \mid \left\{y \in X^k \mid \langle y, k\rangle \in Y\right\} \in N_i^k(x)\right\}.$$

Proposition 4.26. *Suppose that $(X^k, \underline{N}^k)$, $k \in \mathcal{K}$ is a disjoint collection of neighborhood frames and v is a valuation on their disjoint union $(X, \underline{N})$. Then for each $k \in \mathcal{K}$, $v\lceil X^k$ is a valuation on $(X^k, \underline{N}^k)$ which satisfies*

$$(\forall\varphi \in \mathcal{S}(P))\,(\forall x \in X^k)\,\left[(X^k, \underline{N}^k, v) \models_x \varphi \Longleftrightarrow (X, \underline{N}, v) \models_x \varphi\right]$$

and so $\mathcal{L}((X, \underline{N})) = \bigcap_{k\in\mathcal{K}} \mathcal{L}\left((X^k, \underline{N}^k)\right)$.

We can also ask about the counterparts of other familiar concepts surrounding relational semantics. While, as we have seen, 'disjoint union' does interpret naturally in our semantics, and 'frame homomorphism' is present (albeit with a purely algebraic feel) [17], the crucial operation of taking a generated subframe seems to be completely incompatible with neighborhood systems. How about the notion of d-persistence? Can we define such a thing as a 'descriptive neighborhood' frame, and if so would this lead to something which is equivalent to the notion of complexity, as in the relational semantics case? Without these, or similarly natural concepts, the theory of neighborhood frames is unlikely to reach the sophistication of relational frame semantics that we see in works such as ROBERT GOLDBLATT's [32].

We finish off this chapter by noting an interesting relationship between neighborhood frames and relational frames. A logic L_1 is said to simulate a logic L_2 iff there is an appropriate translation t such that for all $\varphi \in \mathcal{S}(P)$, $\varphi \in L_2 \iff t(\varphi) \in L_1$. MARCUS KRACHT and FRANK WOLTER have shown [51] that each monotonic classical logic[6] is complete with respect to its neighborhood frames iff it has a simulation which is complete with respect to the simulating logic's relational frames. KRACHT and WOLTER point out that any normal mono-modal logic falls into this category.

[6] A mono-modal, uni-arity, classical logic L is monotonic iff it satisfies the rule $\varphi \to \psi \in L \Longrightarrow \Box\varphi \to \Box\psi \in L$.

Chapter 5

Non-iterative Logics and Canonicity

5.1 Introduction

This chapter will concern itself with showing that an important class of weak intensional logics is canonical and so in one go we will identify a large volume of logics where canonicity is the norm. Of course, these logics are rather simple and so this result does not have the breadth needed to invalidate the idea that canonicity is an inherently interesting question about intensional logics. This chapter will also highlight the not inconsiderable difference between neighborhood and relational canonicity by producing canonical neighborhood frames which can in no way be related to relational frames.

The logics we look at in this chapter are the 'non-iterative logics' which were first introduced in DAVID LEWIS's 1974 paper [58], where he showed that every intensional logic without iterative axioms has the finite model property. In effect, LEWIS showed that such logics are weakly complete, each non-thesis having a counter-frame. In an unpublished paper ROY A. BENTON [6] conjectured that it should be possible to extend LEWIS's result to show that all such logics are strongly complete, in that every consistent set of formulae can be realised in some frame, possibly even a canonical frame. In this chapter we will show that LEWIS's method can be reinterpreted and augmented to provide a positive answer to this question.

In his book on modal logic [13, p. 261], CHELLAS sets as an exercise, the problem of showing that EK, the equivalential non-normal modal logic formed by taking $\Box(p \to q) \to (\Box p \to \Box q)$ as the only axiom, is canonical.[1] As we have seen, and as CHELLAS had defined it, this means that we must find some neighborhood system over the canonical set of ultrafilters which treats the effable (definable) sets in an appropriate manner. While this gives us considerable

[1] Actually, he sets the problem of showing that EK is determined by its neighborhood frames, however, the method he had in mind was one which used canonical neighborhood frames.

leeway as to which structure we could choose, there is little indication of which non-effable sets can be included in any neighborhood and how they can be stopped from 'destructively' interacting with sets already there. This created a problem with a greater level of difficulty than CHELLAS first assumed. KRISTER SEGERBERG, considered the logic EK in [72],[2] where he commented that there was a completeness result (without proof). Eventually, EK was proved to be canonical in BENTON [6], where an elegantly constructed neighborhood frame was presented, however, this paper never saw print and the result lay forgotten. The logic EK returned to interest recently when it was revisited by CHELLAS and SEGERBERG [15][3] as part of a comprehensive investigation of modal logics in the vicinity of S1. Again, completeness and hence canonicity of these non-iterative logics became a significant issue. This chapter is presented primarily as a solution to the EK problem, however, unlike BENTON's construction, the solution given here applies to all non-iterative logics and the chapter is presented with that in mind.

The reader should be assured that the logic EK and its close relatives are not the only non-normal non-iterative logics to be found in the literature. FRANCIS JEFFRY PELLETIER's proposal for a logic of indeterminacy [64] is an example of a non-iterative logic well away from K.

5.2 Non-Iterative Logics

A number of the early modal logics had a decidedly simplistic structure, in that their axioms do not require that intensional operators interact too strongly with each other. Examples of such axioms are K $\Box(p \to q) \to (\Box p \to \Box q)$, T $\Box p \to p$, D $\Box p \to \Diamond p$, etc., and these axioms have the following common property:

Definition 5.1. A formula $\varphi \in \mathcal{S}(P)$ is *non-iterative* iff every subformula of the form $\Box_i \overline{\psi}$ satisfies the requirement that no intensional operator occurs within any of the ψ_i. Set

$$\mathcal{S}_1(P) = \{\varphi \in \mathcal{S}(\omega) \mid \varphi \text{ is non-iterative}\}.$$

When we go beyond the scope of the simple modal logic of necessity and possibility and start mixing in other intensional operators, we see that the class

[2] SEGERBERG dealt with only two types of canonical objects, the 'canonical model,' and the 'augmented canonical model', so he neither defined nor used CHELLAS's full notion of canonicity.

[3] CHELLAS and SEGERBERG use PK to denote EK, so locating it within a class of classical logics which they call *prenormal*.

of non-iterative formulae includes a large number of well known formulae. LEWIS provided the following representative list of iterative and non-iterative formulae in [58]:

Iterative axioms	*Non-iterative axioms*
$\Diamond p \rightarrow \Box\Diamond p$	$\Box p \rightarrow p$
$(p \longrightarrow\!\!\!\ni q) \rightarrow (\Box p \longrightarrow\!\!\!\ni q)$	$(p \longrightarrow\!\!\!\ni q) \rightarrow (\Box p \rightarrow \Box q)$
$\bigcirc(\bigcirc p \rightarrow p)$	$-(\bigcirc p \,\&\, \bigcirc -p)$
$\mathsf{F}p \rightarrow -\mathsf{P} - \mathsf{F}p$	$\mathsf{F}(p \rightarrow p)$

Using non-iterative formulae as axioms we can produce the non-iterative logics.

Definition 5.2. A logic L is called *non-iterative* iff it can be axiomatised over the rules of modus ponens and replacement of provable equivalents by using only non-iterative formulae and *Taut*.

We should stress here that not all formulae in a non-iterative logic are non-iterative. For example $\Box\Box p \rightarrow \Box p$ is an instance of T.

5.3 Power Set Boolean-Subalgebras

Our goal in this Chapter is to show that all non-iterative logics are canonical and we get a hint at how to solve our problem of extending $(\underline{A}^L, \underline{I}^L)$ to $(\underline{B}^L, \underline{J}^L)$ by looking at the result of LEWIS [58]. Interpreted within the framework of $\mathcal{S}$-algebras, what he showed was:

If $(\underline{A}, \underline{I})$ is an $\mathcal{S}$-algebra and $\underline{B}$ is a power set boolean subalgebra of $\underline{A}$ we can find a $\underline{J}$ which agrees with $\underline{I}$ as much as possible and $(\underline{B}, \underline{J})$ satisfies exactly the same non-iterative formulae as $(\underline{A}, \underline{I})$.

Informally, his argument is:

Let $\underline{B} = \mathcal{P}(X)$, set $J_i(\overline{b}) = I_i(\overline{b}) \cap X$, and note that the non-iterative formulae will be unaffected precisely because they are non-iterative.

In this section we will formalize these ideas and recreate LEWIS's result. We start with the idea of a boolean subalgebra agreeing as much as possible with its parent $\mathcal{S}$-algebra.

Definition 5.3. The algebra $\underline{\underline{B}} = (\underline{B}, \underline{J})$ is a *quasi-$\mathcal{S}$-subalgebra* of $\underline{\underline{A}} = (\underline{A}, \underline{I})$, in symbols $\underline{\underline{B}} \prec \underline{\underline{A}}$, iff

1. $\underline{B}$ is a boolean subalgebra of $\underline{A}$, and
2. $(\forall i \in Idx)\ (\forall \overline{b} \in B)\ [I_i(\overline{b}) \in B \Longrightarrow J_i(\overline{b}) = I_i(\overline{b})]$.

Definition 5.4. We say that $\underline{\underline{B}} = (\underline{B}, \underline{J})$ is a *1-$\mathcal{S}$-subalgebra* of $\underline{\underline{A}} = (\underline{A}, \underline{I})$, in symbols $\underline{\underline{B}} \lhd \underline{\underline{A}}$, iff

1. $\underline{\underline{B}} \prec \underline{\underline{A}}$, and
2. $(\forall \varphi \in \mathcal{S}_1(P)) \left[\underline{\underline{A}} \vDash \varphi \Longrightarrow \underline{\underline{B}} \vDash \varphi\right]$.

So, by descending to a 1-$\mathcal{S}$-subalgebra we are guaranteed to preserve the non-iterative formulae. But as we saw, not all the theses of a non-iterative logic are non-iterative.

Lemma 5.5. *Let L be a non-iterative logic. If $\underline{\underline{B}} \lhd \underline{\underline{A}}$ and $\underline{\underline{A}} \vDash L$ then $\underline{\underline{B}} \vDash L$.*

Proof. $\{\varphi \mid \underline{\underline{B}} \vDash \varphi\}$ is a logic and contains all of L's axioms. □

So now we can formalize and prove LEWIS's observation:

Theorem 5.6. *Let $(\underline{A}, \underline{I})$ be an $\mathcal{S}$-algebra and suppose that $\underline{B}$ is a power set boolean subalgebra of $\underline{A}$. Then there is a $\underline{J}$ such that $(\underline{B}, \underline{J}) \lhd (\underline{A}, \underline{I})$. Moreover, $(\underline{B}, \underline{J})$ satisfies:*

$$(\forall i \in Idx)\,(\forall b, \bar{c} \in B)\,[b \leqslant I_i(\bar{c}) \Longrightarrow b \leqslant J_i(\bar{c})]\,.$$

Proof. We can assume that there is some X such that $\underline{A}$ is a subalgebra of $\underline{\mathcal{P}}(X)$ (for instance, $\underline{A}$ 'is' a subalgebra of the power set of all its ultrafilters). So in particular $A \subseteq \mathcal{P}(X)$.

Now $\underline{B}$ is a power set algebra, so there is an isomorphism $h : \underline{B} \longrightarrow \underline{\mathcal{P}}(Y)$ for some set Y. For each $y \in Y$, $h^{-1}[y]$ is a nonempty set in $\mathcal{P}(X)$, so we can actually take $y \in h^{-1}[y]$, thus without loss of generality we can have $Y \subseteq X$ and $h(b) = \{y \in Y \mid y \in b\} = Y \cap b$. So, define $J_i(\bar{b}) = h^{-1}\left[Y \cap I_i(\bar{b})\right]$.

Since $\underline{B}$ is given as a boolean subalgebra of $\underline{A}$, confirmation that $(\underline{B}, \underline{J}) \lhd (\underline{A}, \underline{I})$ consists of two verifications:

(1) $(\forall i \in Idx)\,(\forall \bar{b} \in B)\left[I_i(\bar{b}) \in B \Longrightarrow J_i(\bar{b}) = I_i(\bar{b})\right]$.

Let $i \in Idx$, $\bar{b} \in B$, and suppose that $b := I_i(\bar{b}) \in B$. Thus

$$J_i(\bar{b}) = h^{-1}\left[Y \cap I_i(\bar{b})\right] = h^{-1}[Y \cap b] = b = I_i(\bar{b})\,.$$

(2) $(\forall \varphi \in \mathcal{S}_1(P))\,[(\underline{A}, \underline{I}) \vDash \varphi \Longrightarrow (\underline{B}, \underline{J}) \vDash \varphi]$.

Let v be a valuation on $(\underline{B}, \underline{J})$, which induces a map $v^B : \mathcal{S}(P) \longrightarrow \underline{B}$. Similarly, v induces a map $v^A : \mathcal{S}(P) \longrightarrow \underline{A}$. Since $\underline{B}$ is a boolean subalgebra of $\underline{A}$, it is a simple induction to show that for ψ a boolean formula, i.e., a formula without intensional operators, $v^A(\psi) = v^B(\psi)$.

Claim: $(\forall y \in Y)\,(\forall \varphi \in \mathcal{S}_1(P))\left[y \in v^B(\varphi) \Longleftrightarrow y \in v^A(\varphi)\right]$.

Proof of claim:

Fix $y \in Y$. The proof of this claim now proceeds by induction on the complexity of φ. The case of $\varphi = p$ is immediate and the inductive steps where φ is a boolean combination are straightforward, so the difficult case is where $\varphi = \Box_i(\overline{\psi})$ and the claim holds for each ψ_i. Remember that $\varphi \in \mathcal{S}_1(P)$ and so each ψ_i does not contain an intensional operator and so is simply a boolean combination of propositional letters. Thus our earlier observation that $v^B(\overline{\psi}) = v^A(\overline{\psi})$ allows us to complete the proof of the claim:

$$\begin{aligned} y \in v^B(\varphi) &\iff y \in v^B(\Box_i(\overline{\psi})) \\ &\iff y \in J_i(v^B(\overline{\psi})) \\ &\iff y \in J_i(v^A(\overline{\psi})) \\ &\iff y \in h^{-1}[Y \cap I_i(v^A(\overline{\psi}))] \\ &\iff y \in I_i(v^A(\overline{\psi}))^{4} \\ &\iff y \in v^A(\Box_i(\overline{\psi})) \\ &\iff y \in v^A(\varphi) \end{aligned}$$

So for $\varphi \in \mathcal{S}_1(P)$ with $(\underline{A}, \underline{I}) \models \varphi$, we have that $v^B(\varphi) = v^A(\varphi) = \top_A = \top_B$. Hence $(\underline{B}, \underline{J}) \models \varphi$. *End of proof of claim.*

We now complete this proof by verifying the last condition mentioned in the conclusion of the theorem.
$(\forall i \in \mathit{Idx})\,(\forall b, \overline{c} \in B)\,[b \leqslant I_i(\overline{c}) \Longrightarrow b \leqslant J_i(\overline{c})]$.

Let $i \in \mathit{Idx}$, $b, \overline{c} \in B$, and suppose that $b \leqslant I_i(\overline{c})$. Then

$$\begin{aligned} J_i(\overline{c}) &= h^{-1}[Y \cap I_i(\overline{c})] \\ &= h^{-1}[(Y \cap I_i(\overline{c}) \cap b) \cup (Y \cap I_i(\overline{c}) \cap \neg b)] \\ &= h^{-1}[(Y \cap I_i(\overline{c}) \cap b)] \cup h^{-1}[Y \cap I_i(\overline{c}) \cap \neg b] \\ &\supseteq h^{-1}[Y \cap I_i(\overline{c}) \cap b] \\ &= h^{-1}[Y \cap b] \\ &= b. \end{aligned}$$

□

To fully recreate LEWIS's result we need to show that we can construct a 1-$\mathcal{S}$-subalgebra of the canonical algebra such that particular non-theses of L

[4] Remember that we have already fixed $y \in Y$.

do not hold in this structure. The following definition and lemma make this precise.

Definition 5.7. Let D be a set, $\underline{\underline{A}}$ an $\mathcal{S}$-algebra. We say that D *denies* φ in $\underline{\underline{A}}$ iff $D \subseteq A$ and there is a valuation v on $\underline{\underline{A}}$ so that for each $\psi <_{subf} \varphi$, $v(\psi) \in D$ and $v(\varphi) \neq \top_A$.

Lemma 5.8. *Let* $\underline{\underline{B}} = (\underline{B}, \underline{J}) \prec (\underline{A}, \underline{I}) = \underline{\underline{A}}$ *and suppose that* $D \subseteq B$ *denies* φ *in* $\underline{\underline{A}}$. *Then* D *denies* φ *in* $(\underline{B}, \underline{J})$.

Proof. Let v be the valuation which witnesses that D denies φ in $\underline{\underline{A}}$. Let v' be a valuation on $\underline{\underline{B}}$ defined by putting $v'(p) = v(p)$ for $p <_{subf} \varphi$ and $v'(p) = \top_B$ otherwise. We make the following claim:
Claim: $(\forall \psi <_{subf} \varphi)\,[v'(\psi) = v(\psi)]$.
Proof of claim:

This is proved by induction on the complexity of ψ. The case of $\psi = p$ is immediate and the inductive case where ψ is a boolean combination is straightforward. Now, in the case where $\psi = \Box_i(\overline{\chi}) <_{subf} \varphi$, with the claim holding for each χ_j, we have that $v(\psi) = I_i(v(\overline{\chi})) \in D \subseteq B$ and $v(\overline{\chi}) \in D \subseteq B$ by D denying φ. Thus by $\underline{\underline{B}} \prec \underline{\underline{A}}$ we can complete the proof of the claim:

$$v(\psi) = I_i(v(\overline{\chi})) = J_i(v(\overline{\chi})) = J_i(v'(\overline{\chi})) = v'(\psi).$$

Thus $(\forall \psi <_{subf} \varphi)\,[v'(\psi) = v(\psi) \in D]$ and $v'(\varphi) = v(\varphi) \neq \top_A = \top_B$. So D denies φ in $\underline{\underline{B}}$. *End of proof of claim.*

□

Now LEWIS's theorem is almost straightforward:

Theorem 5.9 (Lewis). *Every non-iterative logic has the finite frame property and so is weakly complete.*

Proof. Let $\varphi \in \mathcal{S}(P)$ be a non-theorem of L a non-iterative logic and so we have $(\underline{A}^L, \underline{I}^L, v^L) \not\models \varphi$. Let $D^\varphi = \{\overline{v}^L(\psi) \mid \psi <_{subf} \varphi\}$. Trivially D^φ denies φ in $(\underline{A}^L, \underline{I}^L)$. Since D^φ is finite it generates $\underline{B}^\varphi$, a finite boolean subalgebra of $\underline{A}^L$. Every finite boolean subalgebra is a power set boolean algebra and so by Theorem 5.6 there is a $\underline{J}^\varphi$ so that $(\underline{B}^\varphi, \underline{J}^\varphi) \lhd (\underline{A}^L, \underline{I}^L)$. Thus $(\underline{B}^\varphi, \underline{J}^\varphi) \models L$ since L is a non-iterative logic. Moreover, since $D^\varphi \subseteq B^\varphi$ and D^φ denies φ in $(\underline{B}^\varphi, \underline{J}^\varphi)$ we have that $(\underline{B}^\varphi, \underline{J}^\varphi) \not\models \varphi$. Thus $(\underline{B}^\varphi, \underline{J}^\varphi)$ is a finite counter frame for φ.

That L is weakly complete then follows by Proposition 4.5. □

5.4 Canonical Frames

Fix L as an intensional logic. In this section we will complete the construction of a canonical frame for L.

In the previous section we noted that $\underline{B}^{\varphi}$ is a boolean subalgebra of $\underline{A}^L$ and so a $\underline{J}^{\varphi}$ just fell out. Here the analogous approach would be to hope that $\underline{B}^L$ is a boolean subalgebra of $\underline{A}^L$. Unfortunately for us this is not true, but we will see here that it is possible to elementarily extend $(\underline{A}^L, \underline{I}^L)$ to a larger algebra $(\underline{C}^L, \underline{K}^L)$ with the property that $\underline{B}^L$ is a boolean subalgebra of $\underline{C}^L$ and the result will follow.

Our extension from $(\underline{A}^L, \underline{I}^L)$ to $(\underline{C}^L, \underline{K}^L)$ will be through the use of an ultrapower construction and since this is essentially a compactness argument the resulting $\underline{J}^L$ on $\underline{B}^L$ will not be unique nor constructively defined in terms of $(\underline{A}^L, \underline{I}^L)$ and so $(\underline{B}^L, \underline{J}^L)$ can not really lay claim to the title "The Canonical Frame for L." In [6] BENTON produces a particular canonical frame for the logic EK and while this canonical frame has no claim upon being unique it does have the advantage of being constructively defined from $(\underline{A}^L, \underline{I}^L)$.[5]

Before we can attack the main ultrapower argument we need an initial result:

Lemma 5.10. *Let $\underline{D}$ be a finite boolean subalgebra of $\underline{B}^L$. Let $F \subseteq X$ be finite. Then there is a boolean endomorphism $h : \underline{D} \longrightarrow \underline{A}^L$ which is the identity on*[6] *$\underline{D} \cap \underline{A}^L$ and*

$$(\forall x \in F)\,(\forall d \in D)\,[x \in d \iff x \in h(d)]\,.$$

Proof. Let $\underline{C} = \underline{D} \cap \underline{A}^L$, and let

$$\begin{aligned}\mathcal{Y} &= \{c \in C \mid c \text{ is an atom in } \underline{C}\} \text{ and}\\ \mathcal{Z} &= \{d \in D \mid d \text{ is an atom in } \underline{D}\}\,.\end{aligned}$$

Without loss of generality we can assume that F intersects each element of $\mathcal{Z}$ at just one point—otherwise, just increase D so that it includes the singletons $\{x\}$ for $x \in F$.

For each $c \in \mathcal{Y}$ let $\mathcal{Z}_c = \{d \in \mathcal{Z} | d \leqslant c\}$; say $\mathcal{Z}_c = \{d_0^c, \ldots, d_{n_c-1}^c\}$ and so $c = d_0^c \vee \cdots \vee d_{n_c-1}^c$. Recall that each $c \in \mathcal{Y} \subseteq A^L$ is really a set of ultrafilters and since P is infinite, c is an uncountable set. Thus there are distinct points $w_0^c, \ldots, w_{n_c-1}^c \in c$. By our assumption given above we can make our choice so that all elements of F are represented among these ws. By the general theory of canonical models we can find $e_0^c, \ldots, e_{n_c-1}^c \in A^L$ pairwise disjoint sets such that $w_j^c \in e_j^c$ and $c = e_0^c \cup \cdots \cup e_{n_c-1}^c$.

[5] Benton's analysis is in terms of neighborhood frames.

[6] *The boolean algebra $\underline{D} \cap \underline{A}^L$ is the boolean algebra with underlying set $D \cap A^L$.*

Let $\underset{\sim}{\widetilde{D}}$ be the boolean subalgebra of $\underset{\sim}{A}^L$ generated by $\{e_j^c\}_{c\in\mathcal{Y},j<n_c}$. Now, $\{e_j^c\}$ is the set of atoms of $\underset{\sim}{\widetilde{D}}$ since each e_j^c does not meet any distinct $e_k^{c'}$ and $\bigcup_{c,j} e_j^c = \bigcup_c c = \top_A$.

Thus we can define a map $h : \underset{\sim}{D} \longrightarrow \underset{\sim}{\widetilde{D}}$ by setting $h\left(d_j^c\right) = e_j^c, c \in \mathcal{Y}, j < n_c$. This is a 1-1, onto map between the atoms of finite boolean algebras and so h is an isomorphism. Thus $h : \underset{\sim}{D} \longrightarrow \underset{\sim}{A}^L$ is an endomorphism.

To see that h is the identity on $\underset{\sim}{C}$ it is enough to show that it is the identity on $\mathcal{Y}$, the set of atoms of $\underset{\sim}{C}$: Let $c \in \mathcal{Y}$; then:

$$\begin{aligned}
h(c) &= h\left(d_0^c \vee \cdots \vee d_{n_c-1}^c\right) \\
&= h\left(d_0^c\right) \vee \cdots \vee h\left(d_{n_c-1}^c\right) \\
&= e_0^c \vee \cdots \vee e_{n_c-1}^c \\
&= c.
\end{aligned}$$

To finish off the proof note that if $x \in F$ then there is a $c \in \mathcal{Y}$ and some j such that $x = w_j^c$, telling us that

$$\begin{aligned}
x \in d_k^{c'} &\iff c' = c \text{ and } j = k \\
&\iff x \in e_k^{c'} \\
&\iff x \in h(d_k^{c'}).
\end{aligned}$$

Using the homomorphic properties of h we then see that for any $x \in F$ and $d \in D$, $x \in d \iff x \in h(d)$. □

Now we are in a position to deduce that the algebra $\underset{\sim}{B}^L$ is, in some sense, contained within an ultrapower of $\underset{\sim}{A}^L$.

Theorem 5.11. *Let L be an intensional logic.*[7] *Then there is an index Ω, an ultrafilter $\mathcal{U}$ on Ω, and a map $h : \underset{\sim}{B}^L \longrightarrow \underset{\sim}{A}^{L\Omega}/\mathcal{U}$ such that:*

1. *h is a boolean endomorphism into $\underset{\sim}{A}^{L\Omega}/\mathcal{U}$,*
2. *$h\restriction A^L = l$, and*[8]
3. $(\forall u \in \operatorname{ult}(\underset{\sim}{A}^L))\,(\forall b \in B)\,[u^o \in b \iff h(b) \in l(u)]$.

Proof. We will suppress the superscript Ls. Set

$$\Omega = \left\{(\underset{\sim}{D}, F) \mid F \subseteq X^L \text{ and } \underset{\sim}{D} < \underset{\sim}{B} \text{ are both finite}\right\}$$

[7] *There is no restriction, such as non-iterativity, on this logic.*

[8] *Recall that $l : \underset{\approx}{A}^L \longrightarrow \underset{\approx}{A}^{L\Omega}/\mathcal{U}$ is the Łos elementary embedding.*

and, for $(\underline{D}_0, F_0) \in \Omega$, set

$$U(\underline{D}_0, F_0) = \{(\underline{D}, F) \in \Omega \mid D_0 \subseteq D \text{ and } F_0 \subseteq F\}$$

and choose $\mathcal{U}$ to be any ultrafilter which extends

$$\{U(\underline{D}, F) \mid (\underline{D}, F) \in \Omega\}.$$

This set clearly possesses the finite intersection property so we can be assured that it can be extended to an ultrafilter.

For each $\gamma = (\underline{D}, F) \in \Omega$ we can use Lemma 5.10 to give us a map $h_\gamma : \underline{D} \longrightarrow \underline{A}$ which is the identity on $\underline{D} \cap \underline{A}$ and satisfies

$$(\forall x \in F)\,(\forall d \in D)\,[x \in d \Longleftrightarrow x \in h_\gamma(d)]\,.$$

Now, create a map $\widetilde{h} : B \longrightarrow A^\Omega$ by setting $\widetilde{h}(b)(\gamma) = h_\gamma(b)$ and we are able to complete our definitions of $h : \underline{B} \longrightarrow \underline{A}^\Omega/\mathcal{U}$ by setting

$$h(b) = [\widetilde{h}(b)]_{\sim\mathcal{U}}.$$

We then have three conditions to verify:

1. h is a boolean endomorphism into $\underline{A}^\Omega/\mathcal{U}$.

Let $b_1, b_2 \in B$ and let $\underline{D}_0 = \langle\{b_1, b_2\}\rangle$, the boolean subalgebra of $\underline{B}$ generated by $\{b_1, b_2\}$. Let $\gamma_0 = (\underline{D}_0, \varnothing)$ and $\gamma = (\underline{D}, F) \in U(\gamma_0)$. Then

$$\begin{aligned}\widetilde{h}(b_1)(\gamma) \wedge \widetilde{h}(b_2)(\gamma) &= h_\gamma(b_1) \wedge h_\gamma(b_2)\\ &= h_\gamma(b_1 \wedge b_2)\\ &= \widetilde{h}(b_1 \wedge b_2)(\gamma)\end{aligned}$$

where the first and last lines follow because $b_1, b_2 \in D_0 \subseteq D$. Since this held for each $\gamma \in U(\gamma_0) \in \mathcal{U}$ we have that

$$\begin{aligned}h(b_1) \wedge h(b_2) &= [\widetilde{h}(b_1)]_{\sim\mathcal{U}} \wedge [\widetilde{h}(b_2)]_{\sim\mathcal{U}}\\ &= [\widetilde{h}(b_1) \wedge \widetilde{h}(b_2)]_{\sim\mathcal{U}}\\ &= [\widetilde{h}(b_1 \wedge b_2)]_{\sim\mathcal{U}}\\ &= h(b_1 \wedge b_2).\end{aligned}$$

Similarly

$$\begin{aligned}\widetilde{h}(\neg b_1)(\gamma) &= h_\gamma(\neg b_1)\\ &= \neg h_\gamma(b_1)\end{aligned}$$

$$= \neg\widetilde{h}(b_1)(\gamma),$$

and since this holds $\mathcal{U}$ a.e. we have that $h(\neg b_1) = \neg h(b_1)$ allowing us to conclude that h is a boolean homomorphism.

To see that h is one-one note that if $b_1 \neq b_2$ then $h_\gamma(b_1) \neq h_\gamma(b_2)$ since h_γ is one-one on $D \supseteq D_0 \supseteq \{b_1, b_2\}$. So

$$\widetilde{h}(b_1)(\gamma) \neq \widetilde{h}(b_2)(\gamma) \ \mathcal{U} \text{ a.e.}$$

Hence $h(b_1) = [\widetilde{h}(b_1)]_{\sim_\mathcal{U}} \neq [\widetilde{h}(b_2)]_{\sim_\mathcal{U}} = h(b_2)$.

2. $h \restriction A = l$.

Let $a \in A$, $D_0 = \langle\{a\}\rangle$, and $\gamma_0 = (\underline{D}_0, \varnothing)$. We show that $\widetilde{h}(a)(\gamma) = a\,\mathcal{U}$ a.e. by showing that it holds almost everywhere on the set $U(\gamma_0)$.

So let $\gamma = (\underline{D}, F) \in U(\gamma_0)$. Thus h_γ is the identity on $D \cap A \supseteq D_0 \cap A \supseteq \{a\}$ and so $\widetilde{h}(a)(\gamma) = h_\gamma(a) = a$.

Hence $h(a) = [\widetilde{h}(a)]_{\sim_\mathcal{U}} = [\text{const}\,(a)]_{\sim_\mathcal{U}} = l(a)$.

3. $(\forall u \in \text{ult}\,(\underline{A}))\,(\forall b \in B)\,[u^o \in b \iff h(b) \in l(u)]$.

Let $u \in \text{ult}\,(\underline{A})$ and set

$$x = u^o = \{\varphi \in \mathcal{S}\,(P) \mid \|\varphi\| \in u\} \in X^L.$$

Let $b \in B$ and set $\underline{D}_0 = \langle\{b\}\rangle$ and $F_0 = \{x\}$. We show that

$$u^o \in b \iff h_\gamma(b) \in u \ \mathcal{U} \text{ a.e.}$$

by showing that it holds on $U(\underline{D}_0, F_0)$, for if γ is in this set then:

$$\begin{aligned} u^o \in b &\iff x \in b \\ &\iff x \in h_\gamma(b) \\ &\iff h_\gamma(b) \in u. \end{aligned}$$

Now, we finish the proof by noting that,

$$\begin{aligned} h(b) \in l(u) &\iff \widetilde{h}(b)(\gamma) \in u\,\mathcal{U} \text{ a.e.} \\ &\iff h_\gamma(b) \in u\,\mathcal{U} \text{ a.e. on } U(\underline{D}_0, F_0) \\ &\iff u^o \in b. \end{aligned}$$

□

Lemma 5.12. *Let L be an intensional logic.*[9] *Then there is an $\mathcal{S}$-algebra $(\underline{C}^L, \underline{K}^L)$ satisfying*

1. $(\underline{C}^L, \underline{K}^L) \vDash L$,
2. $(\underline{A}^L, \underline{I}^L)$ *is an $\mathcal{S}$-subalgebra of* $(\underline{C}^L, \underline{K}^L)$, *and*
3. $\underline{B}^L$ *is a boolean subalgebra of* $\underline{C}^L$.

Proof. Again, we suppress the superscript Ls. Use Theorem 5.11 to find an Ω, and an ultrafilter $\mathcal{U}$ satisfying the conclusion of that theorem. So $h : \underline{B} \longrightarrow \underline{\underline{A}}^\Omega/\mathcal{U}$ is a homomorphic embedding with $h \restriction A = l$. By Łos's theorem $\underline{\underline{A}}^\Omega/\mathcal{U} \vDash L$, and $l : \underline{\underline{A}} \longrightarrow \underline{\underline{A}}^\Omega/\mathcal{U}$ is also a homomorphic embedding. Thus we can rename the elements $h(b)$ of $h[B]$ to be just b and we will get an algebra $(\underline{C}, \underline{K})$, isomorphic to $\underline{\underline{A}}^\Omega/\mathcal{U}$ which satisfies conditions 1 (by $(\underline{C}, \underline{K})$ being isomorphic to $\underline{\underline{A}}^\Omega/\mathcal{U}$ and $\underline{\underline{A}}^\Omega/\mathcal{U} \vDash L$), 2 (by $h \restriction A = l$ and l being an elementary embedding) and (3 by h being an injective homomorphism). □

This lemma allows us to go on to the desired result that there is a $\underline{J}^L$ so that $(\underline{B}^L, \underline{J}^L)$ really is a canonical frame.

Theorem 5.13. *Let L be a non-iterative intensional logic. Then L is canonical, i.e., there is a $\underline{J}^L$ such that $(\underline{A}^L, \underline{I}^L)$ is a $\mathcal{S}$-subalgebra of $(\underline{B}^L, \underline{J}^L)$ and $(\underline{B}^L, \underline{J}^L) \vDash L$.*

Proof. Let $(\underline{C}^L, \underline{K}^L)$ be as given by Lemma 5.12. $\underline{B}^L$ is a power set boolean subalgebra of $\underline{C}^L$. Thus by Theorem 5.6 there is a $\underline{J}^L$ such that $(\underline{B}^L, \underline{J}^L) \lhd (\underline{C}^L, \underline{K}^L)$. Now, $(\underline{C}^L, \underline{K}^L) \vDash L$ and L is non-iterative, so $(\underline{B}^L, \underline{J}^L) \vDash L$. Thus it remains to show that $(\underline{A}^L, \underline{I}^L)$ is an $\mathcal{S}$-subalgebra of $(\underline{B}^L, \underline{J}^L)$, and all we need to do for this is to show:
$\underline{(\forall i \in Idx)\,(\forall \overline{a} \in A^L)\,[I_i^L(\overline{a}) = J_i^L(\overline{a})]}$.

So let $i \in Idx$ and $\overline{a} \in A^L$. By $(\underline{A}^L, \underline{I}^L)$ a $\mathcal{S}$-subalgebra of $(\underline{C}^L, \underline{K}^L)$ we have that $I_i^L(\overline{a}) = K_i^L(\overline{a})$. This together with $A^L \subseteq B^L$ gives $K_i^L(\overline{a}) \in B^L$ and since $\underline{J}^L$ was constructed such that $(\underline{B}^L, \underline{J}^L) \prec (\underline{C}^L, \underline{K}^L)$ we can conclude that

$$J_k^L(\overline{a}) = K_k^L(\overline{a}) = I_k^L(\overline{a}).$$

□

It is an observation of FRANK WOLTER [102] that the proof given here says a little more about non-iterative logics since our procedure effectively takes an algebra and shows how to construct a power set algebra over its ultrafilters.

[9] *Not necessarily non-iterative.*

This algebra could be called a 'canonical embedding algebra', however unlike the case of normal modal algebras there is no clear recipe for constructing them, and so this concept is inappropriate, or at best may only be available when a logic is provably canonical. We will content ourselves with reporting the apparently weaker extension of Theorem 5.13.

Theorem 5.14. *All non-iterative logics are complex.*

Our proof even shows that every non-iterative logic satisfies the neighborhood equivalent of d-persistence, however it is not clear if this concept is useful, or even natural, in this semantic setting.

5.5 The Normal Modal Logic Case

If we were to confine our interest to the area of normal modal logics our result would be neither new or particularly astounding: We know from LEWIS [58] that all non-iterative logics are weakly complete, from KIT FINE [22] that all elementary normal modal logics are canonical, and from JOHAN VAN BENTHAM [99, p. 99] that all non-iterative normal modal logics are elementary and putting these together gives the result. Unfortunately this argument offers little precedent for the problem of this chapter as use of the canonical accessibility relation is essential, and the argument detours through correspondence theory to allow application of FINE's ingenious result. This argument does produce the full relational result of canonicity whereas our approach will just produce one of the many possible canonical neighborhood frames[10] and may not correspond to the relation on the Stone space.

The proof of Lemma 5.12, in many ways has the 'look and feel' of FINE's result, with the approach of taking frames for the logic, using compactness to mesh them together,[11] and then dropping down to the canonical frame. However, our approach differs markedly in that whereas FINE used compactness to produce emulators of the *saturated points* of a canonical frame, our use of compactness gave us emulators of the unconstrained *sets* of the canonical power set algebra. Whether this is all the similarity that there is, or that there is a deeper correspondence, is open for discussion, and considering these issues may produce a counterpart for the notion of elementarity in relation semantics, which would allow us to emulate the classic proofs of normal modal logic in our more general domain.

[10]Even though the technical discussion of this chapter was with reference to 'algebraic frames', this terminology is not that distinctive and so, as indicated in Chapter 4, for the purposes of this informal discussion we will revert to the term 'neighborhood frame'.

[11]While we use ultraproducts, the result of this chapter can be obtained by compactness—see [93].

We now can consider how the different canonical neighborhood or, equivalently, algebraic frames produced here relate to the underlying canonical relational frame. Remember that the canonical frame production procedure is entirely non-constructive and so there may be many canonical neighborhood frames. We will now graphically illustrate this by presenting *a* canonical neighborhood frame which can in no way have an equivalent relational frame. In fact we will show that under any sensible definition of a relation on this neighborhood frame, the relation will be as small as it can possibly be.

Remember that in this section we are dealing with *normal multi-modal logics*.

We will start by looking at what it means for the neighborhood or, equivalently algebraic, frame to induce a relation on its underlying set of points.

Definition 5.15. Suppose that $(X, \underline{N})$ has an equivalent algebraic frame $(\underline{A}, \underline{I})$, where $\underline{A}$ is just $\underline{\mathcal{P}}(X)$. Then this induces a sequence of binary relations $R(\underline{I})$ on X as follows, for $i \in \mathit{Idx}$:

$$R_i(\underline{I}) := \{\langle x, y\rangle \mid (\forall a \in A)\,[x \in I_i(a) \Longrightarrow y \in a]\}\,.$$

This is of course the natural definition which we would expect given the precedent of Definition 3.32, however it is a little non-standard because it moves from an algebra to its underlying set rather than its set of ultrafilters. It is the definition which would be natural if we were to assume that each neighborhood frame really did correspond directly to a relational frame.

Now we proceed to show that our non-constructive approach to canonical neighborhood frames, or equivalently algebraic frames, does give us quite some leeway. Consider this analog of Lemma 5.12.

Lemma 5.16. *Let L be a normal multi-modal logic.*[12] *Then there is an* $\mathcal{S}$*-algebra* $(\underline{C'}, \underline{K'})$ *satisfying*

1. $(\underline{C'}, \underline{K'}) \vDash L$,
2. $(\underline{A}^L, \underline{I}^L)$ *is an* $\mathcal{S}$*-subalgebra of* $(\underline{C'}, \underline{K'})$,
3. $\underline{B}^L$ *is a boolean subalgebra of* $\underline{C'}$, *and*
4. $(\forall b_1, b_2 \in \mathrm{At}\,\underline{B}^L)\,(\forall i \in \mathit{Idx})\,[b_1 \neq b_2 \Longrightarrow b_1 \leqslant K'_i(\neg b_2)]\,.$

Proof. By Lemma 5.12 there is an $\mathcal{S}$-algebra $(\underline{C}^L, \underline{K}^L)$ satisfying

1. $(\underline{C}^L, \underline{K}^L) \vDash L$,
2. $(\underline{A}^L, \underline{I}^L)$ is an $\mathcal{S}$-subalgebra of $(\underline{C}^L, \underline{K}^L)$, and

[12] *Not necessarily non-iterative.*

3. $\underline{B}^L$ is a boolean subalgebra of $\underline{C}^L$.

To facilitate ease of reading and to avoid clutter we will drop the superscript L throughout. It is also worth reverting to the consideration of $\underline{B}$ as a power-set algebra, $\underline{\mathcal{P}}(X)$.

For each pair $\langle x, y\rangle \in {}^2X$, choose an ultrafilter $\mathcal{U}_y^x$ according to the following prescription:

1. If $y = x$ then set $\mathcal{U}_y^x := \{Y \subseteq X \mid x \in Y\}$.
2. If $y \neq x$ then $\mathcal{U}_y^x$ is any non-principal ultrafilter which extends[13]

$$\{\|\varphi\| \mid \varphi \in y\}.$$

Note here that $\mathcal{U}_y^x$ extends $\{\|\varphi\| \mid \varphi \in y\}$ regardless of whether or not $x \neq y$.

Now for each $x \in X$ define a map $h_x : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, as follows

$$h_x(Y) := \{y \in X \mid Y \in \mathcal{U}_y^x\}.$$

Claim: $(\forall x \in X)\,[h_x : \underline{B} \longrightarrow \underline{B}$ is a boolean homomorphism$]$.

Proof of claim:

Let $x \in X$. We then have 2 verifications:

<u>(1) $(\forall Y, Z \subseteq X)\,[h_x(Y \cap Z) = h_x(Y) \cap h_x(Z)]$.</u>

Let $Y, Z \subseteq X$ and let $y \in X$:

$$\begin{aligned} y \in h_x(Y \cap Z) &\Longleftrightarrow Y \cap Z \in \mathcal{U}_y^x \\ &\Longleftrightarrow Y \in \mathcal{U}_y^x \text{ and } Z \in \mathcal{U}_y^x \\ &\Longleftrightarrow y \in h_x(Y) \text{ and } y \in h_x(Z). \\ &\Longleftrightarrow y \in h_x(Y) \cap h_x(Z). \end{aligned}$$

<u>(2) $(\forall Y \subseteq X)\,[h_x(X - Y) = X - h_x(Y)]$.</u>

Let $Y \subseteq X$ and let $y \in X$:

$$y \in h_x(X - Y) \Longleftrightarrow X - Y \in \mathcal{U}_y^x$$

[13] Such a non-principal ultrafilter can be found since $X - \{y\}$ can be added to this set. For if not then there exist $\varphi_1, \ldots, \varphi_n \in y$ such that $\|\varphi_1\| \cap \cdots \cap \|\varphi_n\| \cap (X - \{y\}) = \varnothing$, so $\|\varphi_1 \wedge \cdots \wedge \varphi_n\| \cap (X - \{y\}) = \varnothing$. But $\varphi := \varphi_1 \wedge \cdots \wedge \varphi_n \in y$, so $\|\varphi\| \cap (X - \{y\}) = \varnothing$, so $y \in \|\varphi\| \subseteq \{y\}$. This tells us that $\|\varphi\| = \{y\}$ and since we have taken our language to be infinite this means that for some p_0 not in the propositional variables of φ, $p_0 \in y$ (or $\neg p_0 \in y$), so $\varphi \to p_0 \in L$ (or $\varphi \to \neg p_0$) so by substitution of $\bot/p_0$ (or $\top/p_0$), $\varphi \to \bot \in L$, thus $\|\varphi\| = \varnothing$, a contradiction.

$$\begin{aligned} &\Longleftrightarrow Y \notin \mathcal{U}_y^x \\ &\Longleftrightarrow y \notin h_x(Y) \\ &\Longleftrightarrow y \in X - h_x(Y). \end{aligned}$$

End of proof of claim.

Claim: $(\forall x \in X)\,[\,h_x \restriction A$ is the identity$]$.

Proof of claim:

We prove the following assertion, for $x \in X$.

$\underline{(\forall \varphi \in \mathcal{S}(P))\,[h_x(\|\varphi\|) = \|\varphi\|]}$.

Let $\varphi \in \mathcal{S}(P)$ and let $y \in X$. Then

$$\begin{aligned} y \in h_x(\|\varphi\|) &\Longleftrightarrow \|\varphi\| \in \mathcal{U}_y^x \\ &\Longleftrightarrow \varphi \in y \\ &\Longleftrightarrow y \in \|\varphi\|, \end{aligned}$$

where the second line follows because $\mathcal{U}_y^x$ extends $\{\|\varphi\| \mid \varphi \in y\}$ which is itself an ultrafilter in the restricted algebra $\underline{A}$.

End of proof of claim.

Thus $h_x \restriction A : (\underline{A}, \underline{I}) \longrightarrow (\underline{A}, \underline{I})$ is an $\mathcal{S}$-algebra isomorphism.

Now define $(\underline{C'}, \underline{K'}) := \prod_{x \in X} (\underline{C}, \underline{K})$ and define $h : \underline{B} \longrightarrow \underline{C'}$ by

$$h(b)(x) = h_x(b).$$

Remember that $h_x : \underline{B} \longrightarrow \underline{B} < \underline{C}$ so each component of h is a homomorphism and thus h is a homomorphism. Moreover each $h_x \restriction A$ is an isomorphism so $h \restriction A$ is an isomorphism of $(\underline{A}, \underline{I})$ to $h[A]$. Moreover

Claim: h is an injection.

Proof of claim:

Let $Y, Z \subseteq X$ and suppose that $Y \neq Z$. Without loss of generality take $y \in Y - Z$. Thus $y \in Y$, so $Y \in \mathcal{U}_y^y$, so $y \in h_y(Y) = h(Y)(y)$. But $y \notin Z$, so $Z \notin \mathcal{U}_y^y$, so $y \notin h_y(Z) = h(Z)(y)$. Thus $h(Y)(y) \neq h(Z)(y)$, allowing us to conclude $h(Y) \neq h(Z)$. *End of proof of claim.*

The function h is an injective boolean homomorphism so, modulo suitable renaming of elements, we can think of

3. $\underline{B}$ [as] a boolean subalgebra of $\underline{C'}$,

and since h can be thought of as the identity on $\underline{A}$, we can think of

2. $(\underline{A}, \underline{I})$ [as] an $\mathcal{S}$-subalgebra of $(\underline{C'}, \underline{K'})$.

Since $(\underline{C'}, \underline{K'})$ is a power of the algebra $(\underline{C}, \underline{K})$ which itself satisfies $(\underline{C}, \underline{K}) \vDash L$ we have that

1. $(\underline{C'}, \underline{K'}) \vDash L$.

This only leaves our last condition which we write as
$\underline{(4)\ (\forall x, y \in X)\ (\forall i \in \mathit{Idx})\ [x \neq y \Longrightarrow x \in K'_i\,(X - \{y\})]}$.

Let $x, y \in X$, $i \in \mathit{Idx}$, and suppose that $x \neq y$. We really want to show that

$$(\forall z \in X)\,[h(\{x\})(z) \subseteq K_i\,(X - h\,(\{y\})\,(z))]$$

which we will rewrite slightly as

$\underline{(\forall z \in X)\,[h_z(\{x\}) \subseteq K_i\,(h_z\,(X - \{y\}))]}$.

Let $z \in X$ and let $w \in h_z\,(\{x\})$. Thus $\{x\} \in \mathcal{U}^z_w$ and we can immediately conclude that $x = z = w$ since this is the only possibility for a smallest element of an ultrafilter. Remember now that $x \neq y$, so $x \in X - \{y\} \in \mathcal{U}^x_x$.

$\underline{h_z\,(X - \{y\}) = X}$

Let $u \in X$ and remember that $z = x$. We have two cases

Case $u = x$

Thus $X - \{y\} \in \mathcal{U}^x_x = \mathcal{U}^x_u$, so $u \in h_z\,(X - \{y\})$.

Case $u \neq x$

Thus $\mathcal{U}^x_u$ is non-principle and so $X - \{y\} \in \mathcal{U}^x_u$ which tells us that $u \in h_z\,(X - \{y\})$.

Hence

$$\begin{aligned} h_z\,(\{x\}) &\subseteq X \\ &= K_i\,(X) \quad \text{by normality} \\ &= K_i\,(h_z\,(X - \{y\}))\,. \end{aligned}$$

□

Remark 5.17. We didn't really need the full force of normality for this result. It would have gone through for any logic that contains the axiom $\Box_i \top$ for each $i \in \mathit{Idx}$, however the value of this result is only in its ability to say something about logics with underlying relations.

We can now move towards our result, however we do need the following definition:

Definition 5.18. Given any set W we define the diagonal relation $\Delta(W)$ as follows:

$$\Delta(W) := \{\langle w, w\rangle \mid w \in W\}.$$

Theorem 5.19. *Let L be a non-iterative normal modal logic. Then L is neighborhood canonical and it has a canonical frame $\left(\underline{B}^L, \underline{J}^L\right)$ where, for all $i \in Idx$,*

$$R_i\left(\underline{J}^L\right) \subseteq \Delta\left(X^L\right).$$

Proof. This proof follows that of Theorem 5.13 in form, however we need to look inside the proofs of Theorem 5.13 and the results it depends on to identify some important elements which we will exploit in this proof.

We start by noting that there is a $(\underline{C}', \underline{K}')$ satisfying conditions 1 through 4 of Lemma 5.16. We use the reasoning of Theorem 5.13 to see that there is a $\underline{J}^L$ such that

$$(\underline{A}^L, \underline{I}^L) < (\underline{B}^L, \underline{J}^L) \triangleleft (\underline{C}', \underline{K}').$$

We must now show that
$(\forall i \in Idx)\left[R_i\left(\underline{J}^L\right) \subseteq \Delta\left(X^L\right)\right]$.

Let $i \in Idx$ and assume that there exists $x, y \in X^L$ with $x \neq y$ and $\langle x, y\rangle \in R_i\left(\underline{J}^L\right)$. Thus by condition 4 in Lemma 5.16 (which was used to produce $(\underline{C}', \underline{K}')$),

$$\{x\} \subseteq K_i'(X - \{y\}).$$

But if we look closely at Theorem 5.6 which gave us the operators $\underline{J}^L$, we have the extra condition on $\underline{J}^L$:

$$\left(\forall b, c \in B^L\right)\left[b \subseteq K_i'(c) \Longrightarrow b \subseteq J_i^L(c)\right].$$

Thus taking $b := \{x\}$ and $c := X - \{y\}$, and our observation above, we can conclude that

$$\{x\} \subseteq J_i^L(X - \{y\}).$$

But this means that $x \in J_i^L(X - \{y\})$ and since $\langle x, y\rangle \in R_i\left(\underline{J}^L\right)$, we can conclude that $y \in X - \{y\}$ a clear contradiction.

□

Remark 5.20. This result cannot be sharpened, for consider any logic L which verifies the axiom T in index i, then if $(\underline{A}, \underline{I}) \vDash L$, $A = \mathcal{P}(W)$, and $\langle w, w\rangle \notin R_i\left(\overline{I}\right)$ we would have that there was some $a \in A$ such that $w \in I_i(a)$ and $w \notin a$. But this contradicts $\Box_i p \to p \in L$.

5.6 Canonicity for Iterative Logics?

While this paper has established canonicity for non-iterative logics, the canonicity, or even completeness of iterative logics, is wide open. To the best of my knowledge there is no result establishing even the finite model property for the logic EK4.[14] The next chapter will be able to answer this question for some iterative logics that we are unable to consider through the techniques of this or earlier chapters. For instance we will be able to show that the McKinsey logic is neighborhood canonical. Unfortunately EK4 still remains elusive and while we will have canonicity if we can demonstrate the finite model property for EK4 we do not know if we even have the finite model property. This makes EK4 an ideal candidate for studying the adequacy of neighborhood semantics for simple non-normal logics. Appendix A details my early thoughts on the behaviour of this system and anyone thinking seriously about the major problem of the completeness of EK4 should consult that Appendix.

[14] Any naïve attempt to carry out a filtration quickly ends in despair because there is no apparent or natural way to determine the action of I on a boolean combination of sets which were forced into our subalgebra.

Chapter 6

Canonicity for Even Logics

6.1 Introduction

In Chapter 5 we saw that every non-iterative intensional logic is neighborhood canonical. In that chapter there were no indications of how one could extend this result to iterative logics. Of course, from Chapters 3 and 4 we have well known relational canonicity results for some normal modal logics that immediately gave us a few results, and we also had the elementary techniques of BRIAN CHELLAS [13] and their refinements given by ROY A. BENTON [6] which gave us others. Unfortunately the normal modal logic results shed no light on the situation with sub-normal modal logics and the elementary techniques of CHELLAS and BENTON only worked for a narrow class of artificially simple logics.

Even the results for normal modal logics had gaps in them. Recall our Conjecture 4.24:

Conjecture 6.1. If a normal modal logic is neighborhood canonical then it is also relational canonical.

A worthwhile logic to pursue with regard to this conjecture is the McKinsey logic KMcK. As noted in Chapter 3 this logic is neither elementary nor canonical—as shown by GOLDBLATT [34]. Could it be that it is neighborhood canonical? It does have the finite frame property (see FINE [21]), and maybe this could be used to construct a canonical neighborhood frame. We could ask ourselves an even easier question about this logic. Recall that WANG [101] has shown that KMcK is not strongly complete for the relational semantics, so what happens when we replace relational frames by neighborhood frames?

In this chapter we advance our understanding of the concept of neighborhood canonicity by identifying a large class of easily recognised logics for which the finite model property will imply neighborhood canonicity. In fact, this class includes the so called *uniform* logics of KIT FINE [21] and so we will quickly get the canonicity of FINE's uniform class—since he showed that these logics have the finite model property. As FINE points out, the McKinsey logic is in this class and so we will have found a counter example to the conjecture above, and then the possibility that the McKinsey logic is not neighborhood strongly complete is also dismissed.

6.2 Even Logics

In this section we shall define what it means for a logic to be even, provide some examples, and compare this notion to that of KIT FINE's uniform logics of [21].

Definition 6.2. We define $\mathcal{S}_e(P) \subseteq \mathcal{S}(P)$, the set of *even* formulae, to be the smallest subset of $\mathcal{S}(P)$ containing

$$\{\Box_i \overline{\varphi} \mid i \in Idx, \overline{\varphi} \in \mathcal{S}_0(P)\}$$

and which is closed under all logical operations.

A logic is called *even* iff it has an axiomatisation that consists entirely of even formulae.

Thus we can easily spot a non-even, or *uneven*, formula just by seeing if it has a subformula which is a boolean combination of intensional formulae and propositional letters. The table below gives examples of formulae which are even and formulae which are uneven—not even.

Even Formulae	*Uneven Formulae*
$\Box(p \to q) \to (\Box p \to \Box q)$	$\Box p \to p$
$\Box p \to \Box\Box p$	$p \to (\Box q \to p)$
$\Box\Diamond p \to \Diamond\Box p$	$\Box(\Box p \to p) \to \Box p$
$\Box\top$	$p \to \Box_+\Diamond_- p$

In [21] FINE defines the following classes—where he dealt with mono-modal normal logics, i.e., *Idx* is of cardinality one, and the arity of $\Box$ is also one.

Definition 6.3 (Fine). We define the sets unif (n), for $n \in \omega$, as follows

$$\text{unif}(0) := \{\Box_i \overline{\varphi} \mid i \in Idx, \overline{\varphi} \in \mathcal{S}_0(P)\},$$

and unif $(n+1)$ is the closure of

$$\{\Box_i \overline{\varphi} \mid i \in Idx, \overline{\varphi} \in \text{unif}(n)\}$$

in $\mathcal{S}(P)$ under the boolean operations. The collection of all uniform formulae is then the set

$$\mathrm{unif}(\omega) := \bigcup_{n\in\omega} \mathrm{unif}(n).$$

A *uniform* logic is one which has a uniform axiomatisation.

Note that the 4 axiom: $\Box p \to \Box\Box p$ is not uniform yet it is even.

As FINE points out, a uniform formula is one where the propositional letters are all nested within a "uniform" number of intensional operators. Clearly this rules out the prospect of having a subformula where a propositional letter is in boolean combination with an intensional operator so we get:

Proposition 6.4. *Every uniform formula is also even and every uniform logic is also even.*

FINE proved [21, p. 234], through a technique of constructing finite models out of the subformulae of a formula itself, the following theorem.

Theorem 6.5 (Fine). *All uniform normal mono-modal logics have the finite model property and so, if finitely axiomatisable, are decidable.*

When FINE refers to the Finite Model Property he is, of course, referring to finite relational frames but these are, of course, a special case of neighborhood frames.

As in the previous chapter we will assume that P is some fixed set of propositional letters, we will suppress the subscript $_P$, and we will then get that all results given in this chapter hold for all infinite sets of propositional letters P.

6.3 Even Pairs

In this section we disclose, well ahead of time, the trick we will use to prove canonicity. Remember that we have to take our power set algebra $\underline{B}^L$ and stick a $\underline{J}^L$ onto it and often it will not be clear exactly how such a $\underline{J}^L$ could be defined. Here we show one technique for generating such a sequence of functions in such a way that the even formulae are preserved. Of course, the conditions necessary for this to go through are relatively restrictive:

Definition 6.6. We say that $\underline{B}$ and $\underline{\underline{A}}$ are *evenly paired* iff:

1. $\underline{B}$ is a boolean algebra,
2. $\underline{\underline{A}}$ is an $\mathcal{S}$-algebra,
3. There is a homomorphism $h : \underline{B} \longrightarrow \underline{A}$,

4. There is a homomorphism $g : \underline{A} \longrightarrow \underline{B}$, and

5. For each $i \in Idx$,

$$(\forall \overline{a} \in A)\, [g(I_i(h \circ g(\overline{a}))) = g(I_i(\overline{a}))] .$$

When this happens, we write $\langle\langle h : \underline{B} \longleftrightarrow \underline{A} : g \rangle\rangle$.

Under these apparently restrictive conditions we can define our $\underline{J}$.

Definition 6.7. The operations $\underline{J}$ on $\underline{B}$ *induced by* $\langle\langle h : \underline{B} \longleftrightarrow (\underline{A}, \underline{I}) : g \rangle\rangle$ are defined as follows: For $i \in Idx$, $\overline{b} \in B$,

$$J_i(\overline{b}) = g(I_i(h(\overline{b}))).$$

Now it is just a matter of verifying, at least as far as even formulae are concerned, that this definition preserves validity.

Theorem 6.8. *Suppose that* $\langle\langle h : \underline{B} \longleftrightarrow (\underline{A}, \underline{I}) : g \rangle\rangle$ *and that* $(\underline{A}, \underline{I}) \models L$ *for* L *an even logic. Then for* $\underline{J}$*, the induced operations on* $\underline{B}$*,*

$$(\underline{B}, \underline{J}) \models L.$$

Proof. Suppose that v is a valuation on $(\underline{B}, \underline{J})$. Define v' a valuation on $(\underline{A}, \underline{I})$ by setting

$$v'(p) = h(v(p)).$$

Thus by h being a boolean homomorphism $v'(\psi) = h(v(\psi))$ for $\psi \in \mathcal{S}_0\,(P)$.
Claim: $(\forall \varphi \in \mathcal{S}_e\,(P))\, [v(\varphi) = g(v'(\varphi))]$.
Proof of claim:

By induction on the complexity of φ as an even formula.
Base Cases: $\varphi = \Box_i \overline{\psi}$ for $\overline{\psi} \in \mathcal{S}_0\,(P)$. Thus

$$\begin{aligned} v(\varphi) &= v(\Box_i \overline{\psi}) \\ &= J_i(v(\overline{\psi})) \\ &= g(I_i(h(v(\overline{\psi})))) \\ &= g(I_i(v'(\overline{\psi}))) \\ &= g(v'(\Box_i \overline{\psi})). \end{aligned}$$

Inductive Hypothesis: Assume that the result holds for the formulae in $\overline{\psi} \in \mathcal{S}_e\,(P)$.
Inductive Step: Suppose that φ is built up from the components of $\overline{\psi}$ in one step.

Case $\varphi = \psi_1 \wedge \psi_2$.

Then we have the equalities:

$$\begin{aligned} v(\varphi) &= v(\psi_1 \wedge \psi_2) \\ &= v(\psi_1) \wedge v(\psi_2) \\ &= g(v'(\psi_1)) \wedge g(v'(\psi_2)) \\ &= g(v'(\psi_1) \wedge v'(\psi_2)) \\ &= g(v'(\psi_1 \wedge \psi_2)) \\ &= g(v'(\varphi)). \end{aligned}$$

Case $\varphi = \neg\psi_1$.

Then we have the equalities:

$$\begin{aligned} v(\varphi) &= v(\neg\psi_1) \\ &= \neg v(\psi_1) \\ &= \neg g(v'(\psi_1)) \\ &= g(\neg v'(\psi_1)) \\ &= g(v'(\neg\psi_1)) \\ &= g(v'(\varphi)). \end{aligned}$$

Case $\varphi = \Box_i\overline{\psi}$.

We get the following sequence of equalities:

$$\begin{aligned} v(\varphi) &= v(\Box_i\overline{\psi}) \\ &= J_i(v(\overline{\psi})) \\ &= g(I_i(h(v(\overline{\psi})))) \\ &= g(I_i(h(g(v'(\overline{\psi}))))) \\ &= g(I_i(v'(\overline{\psi}))) \\ &= g(v'(\Box_i\overline{\psi})) \\ &= g(v'(\varphi)). \end{aligned}$$

Where the fifth line follows because of the fifth condition on $\underline{B}$ and $(\underline{A}, \underline{I})$ being evenly paired.

End of proof of claim.

Now assume that $(\underline{B}, \underline{J}) \not\models L$. Thus there is an even axiom φ and a valuation

v such that $(\underline{B}, \underline{J}, v) \nvDash \varphi$. Let v' be as given above. Hence

$$g(v'(\varphi)) = v(\varphi) \neq \top_B,$$

but since g is a homomorphism $v'(\varphi) \neq \top_A$ and so $(\underline{A}, \underline{I}, v') \nvDash \varphi$ a contradiction to $(\underline{A}, \underline{I}) \vDash L$. □

This gives us the criteria around which the rest of this chapter will be aimed.

6.4 Even Logics with the Finite Model Property

In this section we will slowly work our way through the main proof of this paper, namely that all even logics that have the fmp are neighborhood canonical. So, fix our logic L which we take to be non-trivial, even, and to have the finite model property. Take Ω, $\mathcal{U}$ and h to be the result of applying Theorem 5.11 to the canonical algebras for this logic. Throughout this whole section, we suppress the superscript L, so we will take it as given that $\underline{\underline{A}}$ represents $\underline{\underline{A}}^L$, $\underline{J}$ represents $\underline{J}^L$, etc.

Take $\mathbb{C}$ to be the class of finite $\mathcal{S}$-algebras that verify L and take $\mathbb{F}$ to be the class of all homomorphisms from $\underline{\underline{A}}$ to elements of $\mathbb{C}$.

Proposition 6.9. *Each $f : \underline{\underline{A}} \longrightarrow \underline{\underline{C}}$ in $\mathbb{F}$ has an associated homomorphism*

$$f' : \underline{\underline{A}}^\Omega / \mathcal{U} \longrightarrow \underline{\underline{C}}$$

such that $f' \circ l = f$.

Proof. Let $\underline{\underline{C}}$ and f be as in the hypotheses of this proposition.

Note that since C is finite, for each $\widetilde{a} \in A^\Omega$, there is a $c(\widetilde{a}) \in C$ such that

$$f(\widetilde{a}(\gamma)) = c(\widetilde{a}) \ \mathcal{U} \text{ a.e.}$$

This is because $f(\widetilde{a}(\gamma))$ has only a finite number of possible values, hence one of these values must come up almost all of the time.

For $\widehat{a} \in A^\Omega/\mathcal{U}$ set $f'(\widehat{a}) = c(\operatorname{rep}(\widehat{a}))$ and then it readily follows that f' is a homomorphism and that $f' \circ l = f$. □

Now we observe, using the homomorphisms in $\mathbb{F}$, that every ultrafilter in a finite $\mathcal{S}$-algebra for L will be, in some sense, realised in an ultraproduct of the canonical algebra: Let $\underline{\underline{C}} \in \mathbb{C}$, $f \in \mathbb{F}$, $f : \underline{\underline{A}} \longrightarrow \underline{\underline{C}}$ and let $z \in \operatorname{ult}(\underline{\underline{C}})$. Then $f'^{-1}[z] \in \operatorname{ult}\left(\underline{\underline{A}}^\Omega/\mathcal{U}\right)$.

Definition 6.10. Define *the set of pseudofinite points* to be

$$W = \{f'^{-1}[z] \mid \underline{\underline{C}} \in \mathbb{C},\ z \in \mathrm{ult}\,(\underline{C}),\ f : \underline{\underline{A}} \longrightarrow \underline{\underline{C}} \text{ in } \mathbb{F}\}.$$

Note that W is nonempty as the following argument shows: L is nontrivial, has the finite model property and so there is a $(\underline{C}, \underline{K}) \in \mathbb{C}$ such that $(\underline{C}, \underline{K}) \nvDash \varphi$ for some $\varphi \notin L$. Thus there is a valuation v on $(\underline{C}, \underline{K})$ such that $v(\varphi) \neq \top$. Define a map $f : (\underline{A}, \underline{I}) \longrightarrow (\underline{C}, \underline{K})$ by taking $f(||\psi||) = v(\psi)$.[1] Then $f \in \mathbb{F}$.

Since $\underline{\underline{A}}^{\Omega}/\mathcal{U}$ is an ultrapower of the canonical algebra we could think of its ultrafilters as being sets of formulae in some non-standard, extended language. Then, each element $w \in W$ would correspond to a maximal consistent set in this extended language, and so W would define, in a manner following GROVE [36], a theory $t(W)$ of all, non-standard, formulae which hold throughout W. We will show, relative to the appropriate notions for our algebraic point of view, that every element of X^L can be thought of as being an extension of $t(W)$ and inverting this procedure will give us a map g which will satisfy: $\langle\langle h : \underline{B} \longleftrightarrow \underline{\underline{A}}^{\Omega}/\mathcal{U} : g \rangle\rangle$.

The set W can be thought of as "the collection of points in finite frames for L" and so we can think of $t(W)$ to be "the theory of finite frames for L." So what exactly is $t(W)$?

Definition 6.11. $t(W) = \{\widehat{a} \in A^{\Omega}/\mathcal{U} \mid (\forall w \in W)\,[\widehat{a} \in w]\} = \bigcap W$.

The analog of a theory in a language is a filter in a boolean algebra.

Proposition 6.12. *The set* $t(W)$ *is a proper filter.*

Proof. This consists of a number of verifications. Let $\widehat{a}, \widehat{e} \in A^{\Omega}/\mathcal{U}$.

$\underline{\widehat{a}, \widehat{e} \in t(W) \Longrightarrow \widehat{a} \wedge \widehat{e} \in t(W).}$

Let $\widehat{a}, \widehat{e} \in t(W)$ and $w \in W$. Thus $\widehat{a}, \widehat{e} \in w$ so $\widehat{a} \wedge \widehat{e} \in w$.

$\underline{\widehat{a} \in t(W) \text{ and } \widehat{a} \leqslant \widehat{e} \Longrightarrow \widehat{e} \in t(W).}$

Let $\widehat{a} \in t(W)$, $\widehat{a} \leqslant \widehat{e}$, and let $w \in W$. Thus $\widehat{a} \in w$ and so $\widehat{e} \in w$.

$\underline{\perp \notin t(W).}$

$W \neq \varnothing$ so take $w \in W$. Thus $\perp \notin w \supseteq t(W)$.

□

[1] This is well defined for if $||\psi|| = ||\chi||$ then $\psi \leftrightarrow \chi \in L$ so $v(\psi) = v(\chi)$.

Since $t(W)$, in some sense, represents the class of all finite algebras for L it also has some properties like that of a class of algebras:

Proposition 6.13. *Let $\overline{\overline{a}}, \overline{\overline{e}} \in A^{\Omega}/\mathcal{U}$ and $i \in Idx$. Then*

$$\overline{\overline{a}} \leftrightarrow \overline{\overline{e}} \in t(W) \Longrightarrow \widehat{I}_i(\overline{\overline{a}}) \leftrightarrow \widehat{I}_i(\overline{\overline{e}}) \in t(W).$$

Proof. Suppose that $\overline{\overline{a}} \leftrightarrow \overline{\overline{e}} \in t(W)$, let $w \in W$, and let $(\underline{C}, \underline{K})$, f, z give rise to w. Then $\overline{\overline{a}} \leftrightarrow \overline{\overline{e}} \in f'^{-1}[z']$ for all $z' \in \mathrm{ult}\,(\underline{C})$. Thus

$$\begin{aligned}
(\forall z' \in \mathrm{ult}\,(\underline{C}))\ & \left[f(\overline{\overline{a}} \leftrightarrow \overline{\overline{e}}) \in z'\right] \\
& \Longrightarrow f(\overline{\overline{a}}) \leftrightarrow f(\overline{\overline{e}}) = \top_C \\
& \Longrightarrow K_i(f(\overline{\overline{a}})) \leftrightarrow K_i(f(\overline{\overline{e}})) = \top_C \\
& \Longrightarrow f(K_i(\overline{\overline{a}}) \leftrightarrow K_i(\overline{\overline{e}})) = \top_C \\
& \Longrightarrow f(K_i(\overline{\overline{a}}) \leftrightarrow K_i(\overline{\overline{e}})) \in z \\
& \Longrightarrow K_i(\overline{\overline{a}}) \leftrightarrow K_i(\overline{\overline{e}}) \in w = f'^{-1}[z].
\end{aligned}$$

The first of these clearly holds so we are left with the last and desired result. □

The next two propositions show, in effect, that each $x \in X^L$ can be made into a maximal consistent extension of $t(W)$, which could be thought of as meaning that each $x \in X^L$ 'extends' to a point in a finite frame.

Proposition 6.14. *For all $u \in \mathrm{ult}\,(A)$, the set $l[u]$ has the finite meet property.*

Proof. Let $l(a), l(e) \in l[u]$ for $a, e \in u$. Thus $a \wedge e \in u$ and so $l(a) \wedge l(e) = l(a \wedge e) \in l[u]$. So all we have to show now is that each element of $l[u]$ is non-$\perp$.

Assume that $l(a) = \perp$ for $a \in u$. Thus by l being an elementary embedding (and so one-one) and by $\perp = l(\perp) = l(a) = \perp$ we have that $\perp = a$, so $a \notin u$, a contradiction. So $\perp \notin l[u]$. □

Proposition 6.15. *For all $u \in \mathrm{ult}\,(\underline{A})$, the set $l[u] \cup t(W)$ has the finite meet property.*

Proof. Assume not. Individually, $l[u]$ and $t(W)$ have the finite meet property so there must be an $a \in u$ and an $\widehat{e} \in t(W)$ such that $l(a) \wedge \widehat{e} = \perp$. Thus $\widehat{e} \leqslant \neg l(a)$, so $\neg l(a) \in t(W)$.

But $a \neq \perp$, since $a \in u$, so $\neg a \neq \top$. We can take $a = ||\varphi||$, so $\neg\varphi \notin L$ and thus, by L having the finite model property, there is an algebra $\underline{\underline{C}} \in \mathbb{C}$ for L which denies $\neg\varphi$. Hence there is a valuation v on $\underline{\underline{C}}$ such that $v(\neg\varphi) \neq \top_C$. As before, v induces a homomorphism $f : \underline{\underline{A}} \longrightarrow \underline{\underline{C}}$ such that $f(\neg a) = f(||\neg\varphi||) = v(\neg\varphi) \neq \top_C$.

Take $z \in \mathrm{ult}\,(\underline{C})$ such that $f'(l(\neg a)) = f(\neg a) \notin z$. Set $w = f'^{-1}[z]$ and we see that $l(\neg a) \notin w$. Thus $\neg l(a) \notin w$, a contradiction. □

Let us now pick a function which witnesses the extendability of $x \in X^L$ to maximal consistent extensions of $t(W)$.

Definition 6.16. Define a function $\zeta : \mathrm{ult}\,(\underline{A}) \longrightarrow \mathrm{ult}\,\big(\underline{A}^\Omega/\mathcal{U}\big)$ as follows:

$$\zeta(u) = \begin{cases} l(u) & \text{if } l(u) \in W \\ \text{any ultrafilter extending } l[u] \cup t(W) & \text{otherwise.} \end{cases}$$

We now observe that W is completely covered by ζ.

Proposition 6.17. $\mathrm{range}\,(\zeta) \supseteq W$.

Proof. Let $w \in W$ and suppose that this is witnessed by $\underline{C}$, $f \in \mathbb{F}$, and $z \in \mathrm{ult}\,(\underline{C})$. That is, $w = f'^{-1}[z]$. Set $u = f^{-1}[z]$ and we are done with the following claim (since then $\zeta(u) = l(u) = w$).
Claim: $l(u) = w$.
Proof of claim:
Let $\widehat{a} \in A^\Omega/\mathcal{U}$, and take $\widetilde{a} = \mathrm{rep}\,(\widehat{a})$.

$$\begin{aligned}
\widehat{a} \in l(u) &\iff \widetilde{a}(\gamma) \in u\,\mathcal{U} \text{ a.e.} \\
&\iff \widetilde{a}(\gamma) \in f^{-1}[z]\,\mathcal{U} \text{ a.e.} \\
&\iff f(\widetilde{a}(\gamma)) \in z\,\mathcal{U} \text{ a.e.} \\
&\iff (\exists c \in C)\,[f(\widetilde{a}(\gamma)) = c\,\mathcal{U} \text{ a.e. and } c \in z] \\
&\iff (\exists c \in C)\,[f'(\widehat{a}) = c\,\mathcal{U} \text{ a.e. and } c \in z] \\
&\iff f'(\widehat{a}) \in z\,\mathcal{U} \text{ a.e.} \\
&\iff \widehat{a} \in f'^{-1}[z] = w.
\end{aligned}$$

End of proof of claim.

□

As the 't' function is anti-monotonic and since each $\zeta(u)$ extends $t(W)$ we can conclude:

Corollary 6.18. $t(\mathrm{range}\,(\zeta)) = t(W)$.

We already have the h which was given by Theorem 5.11, so in order to apply Theorem 6.8 we need to produce a g:

Definition 6.19. Take $g : \underline{A}^\Omega/\mathcal{U} \longrightarrow \underline{B}$ to be defined by

$$g(\widehat{a}) = \{x \in X^L \mid \widehat{a} \in \zeta(x^\star)\}.$$

Since $\underline{B}$ is a powerset boolean algebra it is immediate that:

Proposition 6.20. *The map g is a boolean homomorphism.*

Our next result will show that g respects $t(W)$.

Proposition 6.21. $(\forall \widehat{a} \in A^{\Omega}/\mathcal{U})\ [\widehat{a} \in t(W) \Longrightarrow g(\widehat{a}) = \top_B]$.

Proof. Let $\widehat{a} \in A^{\Omega}/\mathcal{U}$ and suppose that $\widehat{a} \in t(W)$. Thus $\widehat{a} \in \bigcap \text{range}\,(\zeta)$, so

$$g(\widehat{a}) = \{x \in X^L \mid \widehat{a} \in \zeta(x^\star)\} = X^L = \top_B.$$

□

To show that we are in a position to apply Theorem 6.8 we only need prove:

Lemma 6.22. $\langle\langle h : \underline{B} \longleftrightarrow \underline{\underline{A}}^{\Omega}/\mathcal{U} : g \rangle\rangle$.

Proof. We just need to verify the last condition of Definition 6.6, which in our case is: For each $i \in \mathit{Idx}$

$$\left(\forall \bar{\widehat{a}} \in A^{\Omega}/\mathcal{U}\right) \left[g(I_i(h \circ g(\bar{\widehat{a}}))) = g(I_i(\bar{\widehat{a}}))\right].$$

We do not need to verify something so complicated since by Proposition 6.13 together with Proposition 6.21 we need only show:

$$(\forall \widehat{a} \in A^{\Omega}/\mathcal{U})\ [h \circ g(\widehat{a}) \leftrightarrow \widehat{a} \in t(W)]\,.$$

So let $\widehat{a} \in A^{\Omega}/\mathcal{U}$, $w \in W$ and let $x \in X^L$ be such that $x^\star = \zeta^{-1}(w)$, so $l(x^\star) = w$. Thus, by the last condition on h given by Theorem 5.11,

$$\begin{aligned} \widehat{a} \in w &\Longleftrightarrow \widehat{a} \in \zeta(x^\star) \\ &\Longleftrightarrow x \in g(\widehat{a}) \\ &\Longleftrightarrow x^{\star o} \in g(\widehat{a}) \\ &\Longleftrightarrow h(g(\widehat{a})) \in l(x^\star) = w. \end{aligned}$$

Thus $\widehat{a} \leftrightarrow h(g(\widehat{a})) \in w$. □

So applying Theorem 6.8 we get a $\underline{J}$ such that $(\underline{B}, \underline{J}) \vDash L$ and canonicity can quickly follow:

Theorem 6.23. *If L is an even logic with the finite model property then L is neighborhood canonical.*

Proof. Since we have that $(\underline{B}, \underline{J}) \vDash L$ we need only verify that $(\underline{A}, \underline{I}) < (\underline{B}, \underline{J})$, and for this we need only show:
$(\forall i \in \mathit{Idx})\ (\forall \bar{a} \in A)\ [J_i(\bar{a}) = I_i(\bar{a})]$.

Let $i \in Idx$ and $\overline{a} \in A$. Then

$$\begin{aligned} J_i(\overline{a}) &= g(\widehat{I}_i(h(\overline{a}))) \\ &= g(\widehat{I}_i(l(\overline{a}))) \\ &= g(l(I_i(\overline{a}))) \\ &= \{x \in X^L \mid l(I_i(\overline{a})) \in \zeta(x^\star)\} \\ &= \{x \in X^L \mid l(I_i(\overline{a})) \in l[x^\star]\} \\ &= \{x \in X^L \mid I_i(\overline{a}) \in x^\star\} \\ &= I_i(\overline{a}), \end{aligned}$$

where the first line follows by Definition 6.7, which gave us our $\underline{J}$, the fifth line by $\zeta(x^\star) \cap l[A] = l[x^\star]$,[2] and the most of the other lines by l being an elementary embedding.

□

As with our result on non-iterative logics we have also essentially shown that each even logic that also has the finite model property is complex.

Another simple extension of this theorem is that if we had another powerset boolean algebra $\underline{B}' = \mathcal{P}(X')$ where $l^{-1}[W] \subseteq X' \subseteq X^L$ then we can also find a suitable $\underline{J}'$ such that $(\underline{B}', \underline{J}') \vDash L$. In some sense, the 'non-finite' points were irrelevant to neighborhood canonicity.

6.5 Conclusion

Our main conclusion, foreshadowed in our introduction, is:

Corollary 6.24. *All uniform normal modal logics are neighborhood canonical.*

Proof. As FINE showed in [21], each logic in this class has the finite model property. Since they are uniform they are also even and the result follows. □

Corollary 6.25. *The logic* KMcK *is neighborhood canonical.*

This answers a question posed in [93].

This chapter has enlarged the class of logics which we know to be neighborhood canonical. This still leaves many gaps since there are many logics which are neither uniform nor have the finite model property. Thus it is natural to ask if there are any other syntactically defined classes like the class of even logics for which similar tricks can be carried out.

[2]The set $\zeta(u)$ extends $l[u]$ which exhaustively bisects $l[A]$.

Also, how important was the assumption that the logics have the finite model property? We needed it because finiteness was essential in taking a homomorphism $f : \underline{\underline{A}} \longrightarrow \underline{\underline{C}}$ and 'extending' it to a homomorphism $f' : \underline{\underline{A}}^{\Omega}/\mathcal{U} \longrightarrow \underline{\underline{C}}$. If we had another class of even logics and associated algebras for which we could do this we would get a similar result. Perhaps this technique could be used to demonstrate that *neighborhood* canonicity in one cardinal implies *neighborhood* canonicity in higher cardinals.

There are many worthwhile uneven logics which have the finite model property and we can ask whether these logics are canonical. Consider, for instance, the provability logics KG and KGrz (The normal logics axiomatised by $\Box(\Box p \to p) \to \Box p$ and $\Box(\Box(p \to \Box p) \to p) \to p$ respectively). These logics are known to have the finite model property yet not be relational canonical. Can we adapt our proof here to show that these logics, too, are neighborhood canonical?

Finally we can ask how our neighborhood constructions relate to the canonical relational constructions. Are there lessons to be learned here for relational canonicity?

Chapter 7

Ultrafilter Semantics and Sahlqvist Logics

7.1 Introduction

Chapters 5 and 6 dealt with neighborhood canonicity while Chapters 8 and 9 will deal with the more traditional notion of canonicity and the fixed notion of a canonical accessibility relation and enclosing frame. This chapter will provide the middle ground. Here we will present an approach to canonicity that will be able to give us a large section of existing and well understood relational canonicity results. At the same time, the technique is perfectly able to handle canonicity for non-iterative non-normal logics. This is a technique that is truly independent of the bounds of normal or, if we consider JÒNSSON's work [43] which has a very similar theme, additive logic.

A result of relative simplicity is that the canonical *model* for a normal modal logic L satisfies L, and, as we saw, the situation becomes more mysterious when we move down to the canonical frame and ask whether this more general structure satisfies L: Some logics fail to be determined by their canonical frame, while other logics are almost trivially determined by their canonical frame. Traditionally, canonicity is proven by first detouring through an analysis of the elementary conditions that the canonical frames satisfy.

Without using the elementary properties of the canonical frame, we cannot get much of an idea of how non-effable sets[1] interact to either verify a logic or reject it. With the canonical model $(X^L, \underline{R}^L, v^L)$ it was easy because each set of interest has the form $\|\varphi\|$ and so just by the fact that $\|\varphi\| = X$ for φ an axiom we conclude canonicity. It would be nice if we could simulate this process for sets not of this form, i.e., if Z is any set then there is some formula φ and some "interpretation" $\mathcal{U}$ such that $Z = \|\varphi\|_{\mathcal{U}}$, where $\|\varphi\|_{\mathcal{U}} = \{x \mid \varphi \in^{\mathcal{U}} x\}$, and which will allow us to conclude $\|\psi\|_{\mathcal{U}} = X$ for ψ an axiom, simply from the fact that

[1]Not of the form $\|\varphi\|$ for some $\varphi \in \mathcal{S}(P)$.

$\psi \in x$ for all x so $\psi \in^{\mathcal{U}} x$. If we can then ensure that (for the most part), for $i \in Idx$, $\|\Box_i\varphi\|_{\mathcal{U}} = I_i(\|\varphi\|_{\mathcal{U}})$, $\|\varphi \wedge \psi\|_{\mathcal{U}} = \|\varphi\|_{\mathcal{U}} \cap \|\psi\|_{\mathcal{U}}$, and $\|\neg\varphi\|_{\mathcal{U}} = X - \|\varphi\|_{\mathcal{U}}$ we would have canonicity.

This chapter will create such a notion of $\|.\|_{\mathcal{U}}$ and will use it to show that the canonicity of any Sahlqvist logic can be derived in this fashion. This will show relational completeness for each of the Sahlqvist logics.

As a result, this is not surprising. Completeness has been known since SAHLQVIST [68] and what is more, Sahlqvist has at least given us a description of the frames for these logics. On a more theoretical level we know that all elementary logics are canonical so, unless the results of this chapter can apply to some logics which are not elementary, we are not going to prove something new. What this chapter does show, however, is that the elementarity part of completeness is unnecessary[2] and it is the author's hope that this chapter sheds some light on the relationship between elementarity and canonicity—particularly in the light of the conjecture that elementarity is equivalent to canonicity.

This chapter does take some time to fully develop all the necessary machinery for Sahlqvist's Theorem, however the flavour of the result can be sampled by just reading to the end of Theorem 7.10 at which point the Ultrafilter Semantics will have been introduced and a motivational theorem will have been proven.

At an appropriate point in our development we will take some time out to show that the technique of "Ultrafilter Semantics" can also handle non-iterative logics and we will rederive, with some prompting from Chapter 5, the results of that chapter. This, more than anything, motivates our interest in this technique as a conceptual tool since it crosses the border separating normal from non-normal logics.

Before continuing we should note that there is a strong similarity between the work reported here, and the algebraic techniques of BJARNI JÒNSSON. In [43] he uses the ideas of JÒNSSON and TARSKI's [44] and an extension by H. RIBEIRO [67] to show that the Sahlqvist logics are canonical. The essence of his work is to analyse the behaviour of closed sets under the action of positive terms.

JÒNSSON's work is part of a small but significant body of results that already exists in the area that could be described as "Sahlqvist without relations" and beyond that is an even larger body of work on extending additive operators to perfect extensions. This chapter was not written as an exposition of a shiny new technique in canonicity theory, rather it is meant to be a bridge between the earlier and later results of this thesis. So, a full analysis in keep-

[2]This is also a result of JÒNSSON's [43].

ing with the existing literature is not within the scope of this work. We will however provide comment in Section 7.9 on the work that has already been carried out in this area and we will indicate how our results fit in.

7.2 Notation

We will adopt a slightly different notation in this chapter so that our treatment of sequences of subsets of the canonical frame will be transparent. In this chapter we use two different sets of propositional letters.

Firstly, we will take L to be a fixed logic and, in all but Section 7.5, we will take L to be a normal modal logic.

We then fix $(X, \underline{R})$ as the canonical frame for L over P, suppressing the subscript P and superscript L.

We have this result about the nature of an R_i image of a point.

Proposition 7.1. *For $x \in X$, $i \in Idx$, $R_i(x)$ is a closed and thus compact set in X.*

Proof. From the definition of $R_i(x)$ we have

$$\begin{aligned} R_i(x) &= \{y \in X \mid \langle x, y\rangle \in R_i\} \\ &= \{y \in X \mid (\forall \varphi \in \mathcal{S}(P))\,[\Box_i\varphi \in x \Longrightarrow \varphi \in y]\} \\ &= \bigcap_{\Box_i\varphi \in x} \|\varphi\|\,. \end{aligned}$$

This is an intersection of clopen sets and so is closed. □

We can now move onto our second set of propositional letters Q. For each $Z \subseteq X$ we will want a representative in Q. Specifically, for each $Z \in \mathcal{P}(X)$, let $d(Z)$ be a new distinct propositional letter. Let $Q := \{d(Z) \mid Z \in \mathcal{P}(X)\}$. Then, of course, $\mathcal{S}(Q)$ is the set of all well formed formulae constructed from Q using the same connectives as before.

A central notational tool of this chapter is that of the $\mathcal{P}(X)$-sequence. Formally, such a sequence is just a function with domain $\mathcal{P}(X)$, but because we will interpret statements about these $\mathcal{P}(X)$-sequences non-standardly we need to distinguish them in our notation.

By writing $\ulcorner e \urcorner$ we will mean a sequence of objects indexed by $\mathcal{P}(X)$,

$$\text{i.e., } \langle e(Z) \mid Z \in \mathcal{P}(X)\rangle.$$

In particular we will adopt the following conventions:

1. If W is some object that has already been fixed (e.g. $\mathcal{S}(P)$) then by $\ulcorner W \urcorner$ we will mean the $\mathcal{P}(X)$-sequence which uniformly takes the value W.

2. As with ordinary finite sequences, if $T(x_0, \dots, x_{n-1})$ is a relation between the values of $\mathcal{P}(X)$-sequences $\ulcorner a_0 \urcorner, \dots, \ulcorner a_{n-1} \urcorner$, then by $T(\ulcorner a_0 \urcorner, \dots, \ulcorner a_{n-1} \urcorner)$ we mean that Q holds "pointwise,"

$$\text{i.e., } (\forall Z \in \mathcal{P}(X))\, T(a_0(Z), \dots, a_{n-1}(Z)).$$

   For instance, by writing

$$\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner \in \ulcorner \mathcal{S}(Q) \urcorner, \text{ and } \vdash_L \ulcorner \varphi \urcorner \to \ulcorner \psi \urcorner$$

   we are requiring that for each $Z \in \mathcal{P}(X)$

$$\varphi(Z), \psi(Z) \in \mathcal{S}(Q), \text{ and } \vdash_L \varphi(Z) \to \psi(Z).$$

3. As with ordinary finite sequences, if W is fixed, $\ulcorner e \urcorner \in \ulcorner W \urcorner$, and $g : W \longrightarrow V$ then by $g(\ulcorner e \urcorner)$ we mean the sequence $\langle g(e(Z)) \mid Z \in \mathcal{P}(X) \rangle$.

4. The $\mathcal{P}(X)$-sequence $\ulcorner D \urcorner \in \ulcorner \mathcal{P}(X) \urcorner$ will be the *diagonal set*, namely:

$$\langle Z \mid Z \in \mathcal{P}(X) \rangle.$$

We will think of elements of $\mathcal{S}(Q)$ as functions of $\mathcal{P}(X)$-sequences, i.e., we write $\varphi \in \mathcal{S}(Q)$ as $\varphi(\ulcorner d \urcorner)$ and $\varphi(\ulcorner \psi \urcorner)$, for $\ulcorner \psi \urcorner \in \ulcorner \mathcal{S}(P) \urcorner$, will mean the element of $\mathcal{S}(P)$ obtained by replacing each occurrence of $d(Z)$ in φ by $\psi(Z)$.

Via the usual interpretation of the connectives $\top, \bot, \wedge, \vee, \neg, \Box_i, \Diamond_i$ as the operations $X, \varnothing, \cap, \cup, \neg, I_i, \check{I}_i$ on $\mathcal{P}(X)$ we get from $\varphi(\ulcorner d \urcorner) \in \mathcal{S}(Q)$ a function

$$\Phi : \ulcorner Z \urcorner \in X \longmapsto \Phi(\ulcorner Z \urcorner).$$

In this way, a logic will be *canonical* iff for all φ in an axiomatisation of L, and all $\ulcorner Z \urcorner \in \ulcorner X \urcorner$ we have that $\Phi(\ulcorner Z \urcorner) = X$—in fact, by L being closed under substitution, it is enough to show that $\Phi(\ulcorner D \urcorner) = X$

We can think of sequences $\ulcorner x \urcorner \in \ulcorner X \urcorner$ as residing in $\prod_{Z \in \mathcal{P}(X)} X$, however as foreshadowed by our notation, it is more convenient to think of them as residing in $\ulcorner X \urcorner$. Further, we will not talk about arbitrary subsets of $\ulcorner X \urcorner$, instead we will abuse the terminology and take a *subset* of $\ulcorner X \urcorner$ to be an object of the form $\ulcorner Z \urcorner$ such that $\ulcorner Z \urcorner \subseteq \ulcorner X \urcorner$. We can go further in our abuse and talk about open and closed subsets:

Definition 7.2. $\ulcorner Z \urcorner \subseteq \ulcorner X \urcorner$ is called *closed* iff for all $Y \in \mathcal{P}(X)$, $Z(Y)$ is closed in X, and *open* (*basic-open*) iff for all $Y \in \mathcal{P}(X)$, $Z(Y)$ is open (basic-open) in X.

Note how all this conforms with our concept of a property of the components of $\ulcorner Z \urcorner$, $\ulcorner X \urcorner$ (in this case closure of one in the other) holding iff it holds pointwise on each component.

We should underline the fact that $\ulcorner Z \urcorner$ is basic-open iff it is of the form $\|\ulcorner \psi \urcorner\|$ for some $\ulcorner \psi \urcorner \in \ulcorner \mathcal{S}(P) \urcorner$.

7.3 Ultrafilter Semantics

In Chapter 8 we will show how non-standard maps between canonical frames can be defined by using an ultrafilter to 'average' out a collection of mappings between the underlying languages. We will use a similar technique here, whereby we use an ultrafilter to tag each set $Z \in \mathcal{P}(X)$ with a propositional letter $d(Z)$ and ensure that $d(Z)$ is appropriately interpreted as Z.

Definition 7.3. Let $\Omega := \left\{ \ulcorner \mu \urcorner \mid \ulcorner \mu \urcorner \in \ulcorner \mathcal{S}(P) \urcorner \right\}$.

Fix, for the moment, $\mathcal{U}$ an ultrafilter on Ω and we will be ready to define $\|.\|_{\mathcal{U}}$, our interpretation of formulae in $\mathcal{S}(Q)$.

Definition 7.4. For each $\varphi(\ulcorner d \urcorner) \in \mathcal{S}(Q)$ let

$$\|\varphi\|_{\mathcal{U}} := \left\{ x \in X \mid \left\{ \ulcorner \mu \urcorner \mid \varphi(\ulcorner \mu \urcorner) \in x \right\} \in \mathcal{U} \right\},$$

i.e., $\|\varphi\|_{\mathcal{U}} = \left\{ x \in X \mid \varphi(\ulcorner \mu \urcorner) \in x \; U \text{ a.e.} \right\}$.

Immediately we can derive a number of simple consequences of this definition:

Proposition 7.5. *For each* $\varphi \in \mathcal{S}(Q)$, $\|\neg\varphi\|_{\mathcal{U}} = \neg\|\varphi\|_{\mathcal{U}}$.

Proof. Let $x \in X$. Then

$$\begin{aligned} x \in \|\neg\varphi\|_{\mathcal{U}} &\iff \neg\varphi(\ulcorner \mu \urcorner) \in x \; \mathcal{U} \text{ a.e.} \\ &\iff \varphi(\ulcorner \mu \urcorner) \notin x \; \mathcal{U} \text{ a.e.} \\ &\iff \textit{not } [\varphi(\ulcorner \mu \urcorner) \in x \; \mathcal{U} \text{ a.e.}] \\ &\iff x \in \neg\|\varphi\|_{\mathcal{U}}. \end{aligned}$$

□

Proposition 7.6. *For all* $\varphi, \psi \in \mathcal{S}(Q)$,

$$\|\varphi \wedge \psi\|_{\mathcal{U}} = \|\varphi\|_{\mathcal{U}} \cap \|\psi\|_{\mathcal{U}}.$$

Proof. Let $x \in X$. Then

$$\begin{aligned} x \in \|\varphi \wedge \psi\|_{\mathcal{U}} &\iff (\varphi \wedge \psi)(\ulcorner \mu \urcorner) \in x\,\mathcal{U} \text{ a.e.} \\ &\iff \varphi(\ulcorner \mu \urcorner) \in x \text{ and } \psi(\ulcorner \mu \urcorner) \in x\,\mathcal{U} \text{ a.e.} \\ &\iff \varphi(\ulcorner \mu \urcorner) \in x\,\mathcal{U} \text{ a.e. and } \psi(\ulcorner \mu \urcorner) \in x\,\mathcal{U} \text{ a.e.} \\ &\iff x \in \|\varphi\|_{\mathcal{U}} \text{ and } x \in \|\psi\|_{\mathcal{U}}. \end{aligned}$$

□

Thus we immediately get the following:

Proposition 7.7. *For all $\varphi, \psi \in \mathcal{S}(Q)$*

1. $\|\varphi \vee \psi\|_{\mathcal{U}} = \|\varphi\|_{\mathcal{U}} \cup \|\psi\|_{\mathcal{U}}$,
2. $\|\varphi \to \psi\|_{\mathcal{U}} = \neg\|\varphi\|_{\mathcal{U}} \cup \|\psi\|_{\mathcal{U}}$,
3. $\|\varphi \leftrightarrow \psi\|_{\mathcal{U}} = \{x \mid x \in \|\varphi\|_{\mathcal{U}} \iff x \in \|\psi\|_{\mathcal{U}}\}$, *and*
4. $\|\varphi \to \psi\|_{\mathcal{U}} = X \implies \|\varphi\|_{\mathcal{U}} \subseteq \|\psi\|_{\mathcal{U}}$.

One other immediate consequence of Definition 7.4 is this crucial result.

Proposition 7.8. *Let $\varphi \in \mathcal{S}(Q)$. Then $\vdash_L \varphi \implies \|\varphi\|_{\mathcal{U}} = X$.*

Proof. Suppose that $\vdash_L \varphi$ and let $x \in X$. Thus $\vdash_L \varphi(\ulcorner \mu \urcorner)$ for all $\ulcorner \mu \urcorner \in \Omega$ since L is closed under substitution. Thus $\varphi(\ulcorner \mu \urcorner) \in x$ for each $\ulcorner \mu \urcorner \in \Omega$ and so $x \in \|\varphi\|_{\mathcal{U}}$. □

Unfortunately things do not run as smoothly when the $\Box_i$ connective is compared with the interior operation I_i.

Proposition 7.9. *Let $\varphi \in \mathcal{S}(Q)$ and $i \in Idx$. Then*

1. $\|\Box_i\varphi\|_{\mathcal{U}} \subseteq I_i(\|\varphi\|_{\mathcal{U}})$ *and*
2. $\|\Diamond\varphi\|_{\mathcal{U}} \supseteq \breve{I}_i(\|\varphi\|_{\mathcal{U}})$.

Proof. **(1)** Let $x \in \|\Box_i\varphi\|_{\mathcal{U}}$. Thus $U := \{\ulcorner \mu \urcorner \mid \Box_i\varphi(\ulcorner \mu \urcorner) \in x\} \in \mathcal{U}$. We want to show that $R_i(x) \subseteq \|\varphi\|_{\mathcal{U}}$. So let $y \in R_i(x)$. For all $\ulcorner \mu \urcorner \in U$, $\Box_i\varphi(\ulcorner \mu \urcorner) \in x$ so $\varphi(\ulcorner \mu \urcorner) \in y$. Thus $\varphi(\ulcorner \mu \urcorner) \in y\,\mathcal{U}$ a.e. So $y \in \|\varphi\|_{\mathcal{U}}$.
(2) Follows from (1) by duality. □

The reverse inclusions do not usually hold because it would amount to requiring that $\mathcal{U}$ is closed under (certain) infinite intersections. We will, however, construct a $\mathcal{U}$ so that the reverse inclusions do hold for certain φ.

As a prelude for what is in store we are now in a position to prove a simple canonicity result. This will give the flavour of the arguments to come and readers are asked to keep the argument given here in mind as our our full result is nothing but a generalisation of this.

Theorem 7.10. *Let $m, n \in \omega$, $i, j \in Idx$. Suppose that $\Diamond_i^n p \to \Box_j{}^m p \in L$. Then $(X, R) \models \Diamond_i^n p \to \Box_j{}^m p$.*

Proof. Let $Z \subseteq X$ and let $p = d(Z)$. We need to show that $\breve{I}_i^n(Z) \subseteq I_j^m(Z)$. For each finite $F \subseteq X$ let

$$U(F) = \{\ulcorner \mu \urcorner \in \Omega \mid (\forall x \in F)\,[\mu(Z) \in x \Longleftrightarrow x \in Z]\}\,.$$

From this definition it can readily be seen that

$$(\forall F_1, F_2 \subseteq X \text{ finite})\,[U(F_1 \cup F_2) \subseteq U(F_1) \cap U(F_2)]$$

so $\{U(F) \mid F \subseteq X \text{ finite}\}$ will have the finite intersection property if we can show:

Claim: $(\forall F \subseteq X \text{ finite})\,[U(F) \neq \varnothing]$

Proof of claim:

Let $F \subseteq X$ be finite. Since F is finite and we are in a Hausdorff space X we can find basic open sets $\langle \|\varphi_x\| \mid x \in F\rangle$ such that

$$(\forall x, y \in F)\,[x \in \|\varphi_y\| \Longleftrightarrow x = y]\,.$$

Define $\ulcorner \mu \urcorner$ to be any $\mathcal{P}(X)$-sequence which satisfies

$$\mu(Z) = \bigvee_{x \in F \cap Z} \varphi_x.$$

We must verify that
$\underline{(\forall x \in F)\,[p(\ulcorner \mu \urcorner) \in x \Longleftrightarrow x \in Z]}$

Let $x \in F$

$$\begin{aligned} p(\ulcorner \mu \urcorner) \in x \Longleftrightarrow \mu(Z) \in x &\Longleftrightarrow \bigvee_{y \in F \cap Z} \varphi_y \in x \\ &\Longleftrightarrow (\exists y \in F \cap Z)\,[\varphi_y \in x] \\ &\Longleftrightarrow (\exists y \in F \cap Z)\,[x = y] \\ &\Longleftrightarrow x \in Z. \end{aligned}$$

So $\ulcorner \mu \urcorner \in U(F) \neq \varnothing$. *End of proof of claim.*

So let $\mathcal{U}$ extend $\{U(F) \mid F \subseteq X \text{ finite}\}$.

Claim: $\|p\|_{\mathcal{U}} = Z$
Proof of claim:

($\subseteq$) Let $x \in Z$. Thus $(\forall^{\ulcorner}\mu^{\urcorner} \in U(\{x\}))\ [p(^{\ulcorner}\mu^{\urcorner}) \in x]$ and so $p(^{\ulcorner}\mu^{\urcorner}) \in x\ \mathcal{U}$ a.e. giving $x \in \|p\|_{\mathcal{U}}$.

($\supseteq$) Let $x \in \neg Z$. Thus

$$(\forall^{\ulcorner}\mu^{\urcorner} \in U(\{x\}))\ [p(^{\ulcorner}\mu^{\urcorner}) \notin x]$$

and so *not* $[p(^{\ulcorner}\mu^{\urcorner}) \in x\ \mathcal{U}$ a.e. $]$ giving $x \notin \|p\|_{\mathcal{U}}$. *End of proof of claim.*

Now, the canonicity condition is easy to verify:

$$\begin{aligned} J_i^n(Z) &= J_i^n(\|p\|_{\mathcal{U}}) \\ &\subseteq \|\Diamond_i^n p\|_{\mathcal{U}} \\ &\subseteq \|\Box_j{}^m p\|_{\mathcal{U}} \\ &\subseteq I_j^m(\|p\|_{\mathcal{U}}) \\ &= I_j^m(Z) \end{aligned}$$

With the first line following from the second claim, the second and fourth lines from Proposition 7.9, and the third line from $\vdash_L \Diamond^n p \to \Box_i{}^m p$ and Proposition 7.8 (4) and Proposition 7.7. □

We have proved that these (obviously elementary) logics are canonical and so complete with respect to their relational semantics. Unfortunately the proof sheds no light on the nature of the logics' relational frames, however the proof is entirely by basic properties of the canonical model and depended only on the fact that we can express Z as $\|d(Z)\|_{\mathcal{U}}$.

In subsequent sections we will refine this approach and show that we can provide "instant" canonicity for all the Sahlqvist logics.

7.4 Combining Ultrafilters

Before proceeding we can save some conceptual effort later on by discussion how ultrafilters can be 'combined.' This will of course be based on how we combine elements of ultrafilters and, even more fundamentally, on how we combine elements of their elements.

Definition 7.11. For $^{\ulcorner}\mu_1{}^{\urcorner}, {}^{\ulcorner}\mu_2{}^{\urcorner} \in \Omega$ define $^{\ulcorner}\mu_1 \wedge \mu_2{}^{\urcorner} \in \Omega$ as follows:

$$(\forall Z \in \mathcal{P}(X))\,[(\mu_1 \wedge \mu_2)(Z) = \mu_1(Z) \wedge \mu_2(Z)].$$

In this way we can define $^{\ulcorner}\bigwedge_{\alpha \in F} \mu_\alpha{}^{\urcorner}$ for F a finite set and $^{\ulcorner}\mu_\alpha{}^{\urcorner} \in \Omega$, for $\alpha \in F$.

We deliberately ignore issues raised by different orderings of F.

Definition 7.12. Let $\vec{U} = \langle U_\alpha \mid \alpha \in F\rangle$ be a finite sequence in $\mathcal{P}(\Omega)$. Define:

$$\bigotimes \vec{U} := \left\{ \ulcorner \bigwedge_{\alpha \in F} \mu_\alpha \urcorner \mid (\forall \alpha \in F) \left[\ulcorner \mu_\alpha \urcorner \in U_\alpha \right] \right\}.$$

This will not work for $\vec{U}$ an infinite sequence, so we need the services of an auxiliary ultrafilter.

Start with any set G. Let $\Gamma(G) := \{F \mid F \subseteq G \text{ is finite}\}$. For each $F_0 \in \Gamma(G)$ let $\Delta(F_0) \subseteq \mathcal{P}(\Gamma(G))$ be defined by

$$\Delta(F_0) := \{F \in \Gamma(G) \mid F \supseteq F_0\}.$$

Clearly $\{\Delta(F) \mid F \in \Gamma(G)\}$ has the fip since $F \in \Delta(F)$ and $\Delta(F_1) \cap \Delta(F_2) = \Delta(F_1 \cup F_2)$. So it can be extended to an ultrafilter $\nabla(G)$. We will take this ultrafilter to be fixed for each set G.

Definition 7.13. Let $\vec{U} = \langle U_\alpha \mid \alpha \in G\rangle$ be any sequence in $\mathcal{P}(\Omega)$. For $\Delta \in \nabla(G)$ set

$$\bigotimes_{\Delta} \vec{U} := \bigcup_{F \in \Delta} \bigotimes \langle U_\alpha \mid \alpha \in F\rangle.$$

Now we can define the 'conjunction' of a sequence of ultrafilters:

Definition 7.14. Let $\vec{\mathcal{U}} = \langle \mathcal{U}_\alpha \mid \alpha \in G\rangle$ be any sequence of ultrafilters over $\mathcal{P}(\Omega)$. Let $\bigotimes \vec{\mathcal{U}}$ be a (fixed) ultrafilter which extends

$$\left\{ \bigotimes_{\Delta} \vec{U} \mid (\forall \alpha \in G) \left[U_\alpha \in \mathcal{U}_\alpha \right], \Delta \in \nabla(G) \right\},$$

a set that clearly has the finite intersection property.

This conjunction construction will come in useful once we have introduced the various types of positive formulae in Section 7.6.

7.5 Non-iterative Logics Revisited

We will take a break now from our progression towards Sahlqvist's theorem to show that our approach does unify the normal and non-normal. That is, we will use the technique of ultrafilter semantics to rederive the canonicity of non-iterative classical logics.

For this section only, take L to be any classical multi-modal, multi-adic modal logic.

In obtaining our canonicity result we will decide whether to put $\overline{Z}$ into $N(x)$ by looking at whether $x \in \left\|\Box_i d(\overline{Z})\right\|_{\mathcal{U}}$. Of course, we need to work a little to show that there really is a $\mathcal{U}$ that will satisfy the canonicity condition and get rid of ambiguity.

For each finite $G \subseteq \mathcal{P}(X)$, let $\underline{D}(G)$ be the finite boolean subalgebra of $\underline{B}^L$ generated by G and let $\underline{C}(G) = \underline{D}(G) \cap \underline{A}^L$.

Each $\ulcorner \mu \urcorner \in \Omega$ induces a map $\lambda(\ulcorner \mu \urcorner) : B^L \longrightarrow A^L$ defined by $\lambda(\ulcorner \mu \urcorner)(Z) = \|d(Z)(\ulcorner \mu \urcorner)\|$.

We can now start the process of defining our ultrafilter $\mathcal{U}$. For a finite $F \subseteq X$, and finite $G \subseteq \mathcal{P}(X)$ let $U(F,G)$ be the set

$$\begin{aligned}\{\ulcorner \mu \urcorner \in \Omega \mid & \lambda(\ulcorner \mu \urcorner) \text{ is a one-one homomorphism on } \underline{D}(G) \\ & \text{which is the identity on } \underline{C}(G) \text{ and} \\ & \text{for each } x \in F \text{ and each } Z \in G, x \in Z \Longleftrightarrow x \in \lambda(\ulcorner \mu \urcorner)(Z)\}.\end{aligned}$$

Lemma 7.15. *For finite $F_1, F_2 \subseteq X$ and finite $G_1, G_2 \subseteq \mathcal{P}(X)$*

1. $U(F_1 \cup F_2, G_1 \cup G_2) \subseteq U(F_1, G_1) \cap U(F_2, G_2)$, *and*
2. $U(F_1, G_1) \neq \varnothing$.

Proof. (1). This follows easily from the fact that each of the defining properties of $U(F,G)$ is closed under subsets.

For (2), Lemma 5.10 guarantees the existence of a homomorphism h which satisfies the properties of $\lambda(\ulcorner \mu \urcorner)$. For any $Z \in G$, $h(Z) \in A$ so set $d(Z)(\ulcorner \mu \urcorner) = \varphi$ where $\|\varphi\| = h(Z)$. □

From this Lemma we see that

$$\{U(F,G) \mid F \subseteq X \text{ and } G \subseteq \mathcal{P}(X) \text{ are finite}\}$$

can be extended to an ultrafilter $\mathcal{U}$.

Lemma 7.16. *For all $\overline{\varphi} \in \mathcal{S}(P)$, and all $i \in Idx$, $\|\Box_i d(\|\overline{\varphi}\|)\|_{\mathcal{U}} = \|\Box_i \overline{\varphi}\|$.*

Proof. Let $\varphi \in \mathcal{S}(P)$, $\overline{Z} := \|\overline{\varphi}\|$, $F := \varnothing$, and $G := \{Z\}$. For all $\ulcorner \mu \urcorner \in U(F,G)$, $\lambda(\ulcorner \mu \urcorner)$ is the identity on $D(G) \cap A \supseteq \{Z\}$. Thus

$$\left\|d(\overline{Z})(\ulcorner \mu \urcorner)\right\| = \lambda(\ulcorner \mu \urcorner)(\overline{Z}) = \overline{Z} = \|\overline{\varphi}\|\,.$$

This tells us that $\vdash_L d(\overline{Z})(\ulcorner \mu \urcorner) \leftrightarrow \overline{\varphi}$ and so by the fact that L is at least a classical logic $\vdash_L \Box_i d(\overline{Z})(\ulcorner \mu \urcorner) \leftrightarrow \Box_i \overline{\varphi}$, which means that $\left\|\Box_i d(\overline{Z})(\ulcorner \mu \urcorner)\right\| = \|\Box_i \overline{\varphi}\|$.

We have just shown that $x \in \left\|\Box_i d(\overline{Z})(\ulcorner\mu\urcorner)\right\| \mathcal{U}$ a.e. iff $x \in \|\Box_i\overline{\varphi}\|_{\mathcal{U}}$, i.e., $x \in \left\|\Box_i d(\overline{Z})\right\|_{\mathcal{U}}$ iff $x \in \|\Box_i\overline{\varphi}\|_{\mathcal{U}}$ as required. □

Now we can see that $\mathcal{U}$ ensures that $d(Z)$ really does represent Z.

Lemma 7.17. $(\forall Z \in \mathcal{P}(X))\,[\|d(Z)\|_{\mathcal{U}} = Z]$.

Proof. Let $x \in X$, and let $\ulcorner\mu\urcorner \in U(\{x\}, \{Z\})$. Thus

$$x \in Z \iff d(Z)(\ulcorner\mu\urcorner) \in x.$$

Since this held for any $\ulcorner\mu\urcorner$ we have that

$$x \in Z \iff (\forall\ulcorner\mu\urcorner \in U(\{x\}, \{Z\}))\,[d(Z)(\ulcorner\mu\urcorner) \in x]$$

By $U(\{x\}, \{Z\}) \in \mathcal{U}$ we get that

$$x \in Z \iff x \in \|d(Z)\|_{\mathcal{U}}.$$

□

Now, for $i \in Idx$, define $N(x) := \left\{\overline{Z} \mid x \in \left\|\Box_i d(\overline{Z})\right\|_{\mathcal{U}}\right\}$. Thus $I_i(Z) = \|\Box_i d(Z)\|_{\mathcal{U}}$.

We must verify that $\underline{N}$ satisfies the first condition of a neighborhood canonical frame that is given in Chapter 4

Lemma 7.18. *For all* $\varphi \in \mathcal{S}(P)$ *and* $i \in Idx$,

$$\|\overline{\varphi}\| \in N_i(x) \iff \Box_i\varphi \in x.$$

Proof. This is just a restatement of Lemma 7.16:

$$\begin{aligned}\|\overline{\varphi}\| \in N_i(x) &\iff x \in \|\Box_i d(\|\overline{\varphi}\|)\|_{\mathcal{U}} = \|\Box_i\varphi\| \\ &\iff \Box_i\overline{\varphi} \in x.\end{aligned}$$

□

As a final lemma before the main theorem of this section, we need to verify that $\|\varphi\|_{\mathcal{U}}$ really is representative of what it claims to represent—at least for non-iterative formulae.

Lemma 7.19. *Let* $\varphi \in \mathcal{S}_1(Q)$ *have propositional variables among* $\langle d(Z_k)\rangle_{k<n}$. *Then*

$$\|\varphi\|_{\mathcal{U}} = \Phi\left(\overline{Z}\right).$$

Proof. We proceed by induction on the complexity of φ.

The base case follows by Lemma 7.17.

The Boolean steps follow by Propositions 7.6 and 7.5.

The only remaining step is when we deal with $\Box_i\overline{\varphi}$ for $\overline{\varphi} \in \mathcal{S}_0(Q)$. We know that $\|\overline{\varphi}\|_{\mathcal{U}} = \overline{\Phi}(\overline{Z})$ by the inductive hypothesis. Let $\overline{W} = \overline{\Phi}(\overline{Z})$, and we must show that $I_i(\overline{W}) = \|\Box_i\overline{\varphi}\|_{\mathcal{U}}$.

So let $\ulcorner\mu\urcorner \in U(\varnothing, \{Z_0, \ldots, Z_{n-1}\} \cup \{W_0, \ldots, W_{n_i-1}\})$. Since $\lambda(\ulcorner\mu\urcorner)$ is a boolean homomorphism we get that $\lambda(\ulcorner\mu\urcorner)(\overline{W}) = \overline{\Phi}(\lambda(\ulcorner\mu\urcorner)(\overline{Z}))$, telling us that

$$\vdash_L d(\overline{W})(\ulcorner\mu\urcorner) \leftrightarrow \overline{\varphi}(\ulcorner\mu\urcorner)$$

and so

$$\vdash_L \Box_i d(\overline{W})(\ulcorner\mu\urcorner) \leftrightarrow \Box_i\overline{\varphi}(\ulcorner\mu\urcorner).$$

Hence $I_i(\overline{W}) = \|\Box_i d(Z_0)(\ulcorner\mu\urcorner)\|_{\mathcal{U}} = \|\Box_i\overline{\varphi}\|_{\mathcal{U}}$ as required. □

Now we are able to provide a canonicity result.

Theorem 7.20. *All non-iterative logics are neighborhood canonical.*

Proof. Suppose that L is a non-iterative logic. Let $\varphi(\overline{p}) \in \mathcal{S}(P)$ be a non-iterative axiom of L and let $\overline{Z} \in \mathcal{P}(X)$. We must show that

$$\Phi(\overline{Z}) = X.$$

Let ψ result from φ by replacing p_j by $d(Z_j)$ for $j < n$. Thus

$$\Phi(\overline{Z}) = \Psi(\overline{Z}) = \|\psi\|_{\mathcal{U}} = X$$

by identity, Lemma 7.19, and Proposition 7.8. □

7.6 Positive Formulae

We now return from our sojourn below normality to the realm of uni-adic multi-modal logics.

In the derivation of the non-elementary version of Sahlqvist's Theorem we will need to take a look at its attendant formula types with a view to our "ultrafilter analysis." In particular we need the following definitions:

Definition 7.21. A formula is the dual of a formula φ if it is the result of replacing each occurrence of $\top$, $\bot$, $\wedge$, $\vee$, $\Box_i$, and $\Diamond_i$ in φ with their duals $\bot$, $\top$, $\vee$, $\wedge$, $\Diamond_i$, and $\Box_i$ respectively. We denote this formula by $\breve{\varphi}$.

Definition 7.22. We call a formula φ *positive* iff it is built up from propositional letters using only $\top$, $\bot$, $\wedge$, $\vee$, $\Box_i$, and $\Diamond_i$. We call a positive formula *strongly-positive* if it contains no $\Diamond_i$s for all $i \in Idx$.

Notice that the dual of a positive formula is still a positive formula.

Strongly positive formulae are particularly friendly to ultrafilters as the following result shows.

Proposition 7.23. *If $\varphi \in \mathcal{S}(Q)$ is strongly-positive and $\ulcorner\psi\urcorner \in \ulcorner\mathcal{S}(Q)\urcorner$ then*

$$\|\varphi(\ulcorner\psi\urcorner)\|_u \subseteq \Phi(\|\ulcorner\psi\urcorner\|_u).$$

Proof. By induction on the complexity of φ.
Base Cases: If φ is in Q or is $\top$ or $\bot$ then the result holds immediately.
Inductive Hypothesis: The result holds for strongly-positive $\varphi_1, \varphi_2 \in \mathcal{S}(Q)$.
Inductive Step: Let $\ulcorner\psi\urcorner \in \ulcorner\mathcal{S}(Q)\urcorner$. We want to show that the result holds for φ.

Case $\varphi = \varphi_1 \wedge \varphi_2$

Then

$$\begin{aligned}\|\varphi(\ulcorner\psi\urcorner)\|_u &= \|\varphi_1(\ulcorner\psi\urcorner)\|_u \cap \|\varphi_2(\ulcorner\psi\urcorner)\|_u \\ &\subseteq \Phi_1(\|\ulcorner\psi\urcorner\|_u) \cap \Phi_2(\|\ulcorner\psi\urcorner\|_u) \\ &\subseteq \Phi(\|\ulcorner\psi\urcorner\|_u).\end{aligned}$$

Case $\varphi = \varphi_1 \vee \varphi_2$

Then

$$\begin{aligned}\|\varphi(\ulcorner\psi\urcorner)\|_u &= \|\varphi_1(\ulcorner\psi\urcorner)\|_u \cup \|\varphi_2(\ulcorner\psi\urcorner)\|_u \\ &\subseteq \Phi_1(\|\ulcorner\psi\urcorner\|_u) \cup \Phi_2(\|\ulcorner\psi\urcorner\|_u) \\ &\subseteq \Phi(\|\ulcorner\psi\urcorner\|_u).\end{aligned}$$

Case $\varphi = \Box_i\varphi_1$

Then

$$\begin{aligned}\|\varphi(\ulcorner\psi\urcorner)\|_u &= \|\Box_i\varphi_1(\ulcorner\psi\urcorner)\|_u \\ &\subseteq I_i(\|\varphi_1(\ulcorner\psi\urcorner)\|_u) \\ &\subseteq I_i(\Phi_1(\|\ulcorner\psi\urcorner\|_u)) \\ &= \Phi(\|\ulcorner\psi\urcorner\|_u).\end{aligned}$$

With the second line following from Proposition 7.9 and the third line from the monotonicity of I.

$\square$

To deal with the antecedent part of Sahlqvist formulae we will need the following concept.

Definition 7.24. A positive formula φ is *dual-strongly-positive* if it contains no mention of $\square_i$, i.e., it is a dual of a strongly-positive formula.

By duality, or induction on complexity, we can readily obtain that:

Proposition 7.25. *If $\varphi \in \mathcal{S}(Q)$ is dual-strongly-positive and $\ulcorner\psi\urcorner \in \ulcorner\mathcal{S}(Q)\urcorner$ then*

$$\|\varphi(\ulcorner\psi\urcorner)\|_u \supseteq \Phi(\|\ulcorner\psi\urcorner\|_u).$$

Because of the presence of $\Diamond_i$s in arbitrary positive formulae we cannot extend Proposition 7.23 to them. Positive formulae are, however, monotonically increasing in each variable. More precisely:

Proposition 7.26. *Let $\varphi \in \mathcal{S}(Q)$ be a positive formula. Then*

1. $\left(\forall\ulcorner\psi\urcorner,\ulcorner\chi\urcorner \in \ulcorner\mathcal{S}(Q)\urcorner\right)\left[\vdash_L \ulcorner\psi\urcorner \to \ulcorner\chi\urcorner \Longrightarrow \vdash_L \varphi(\ulcorner\psi\urcorner) \to \varphi(\ulcorner\chi\urcorner)\right]$ *and*

2. $\left(\forall\ulcorner Z\urcorner,\ulcorner W\urcorner \in \ulcorner\mathcal{P}(X)\urcorner\right)\left[\ulcorner Z\urcorner \subseteq \ulcorner W\urcorner \Longrightarrow \Phi(\ulcorner Z\urcorner) \subseteq \Phi(\ulcorner W\urcorner)\right].$

Proof. By induction on the complexity of φ. We will just prove (1) above as (2) follows in a similar fashion.
Base Cases: If φ is in Q or is $\top$ or $\bot$ then (1) holds immediately.
Inductive Hypothesis: That (1) holds for positive $\varphi_1, \varphi_2 \in \mathcal{S}(Q)$.
Inductive Step: Let $\ulcorner\psi\urcorner, \ulcorner\chi\urcorner \in \ulcorner\mathcal{S}(Q)\urcorner$ and suppose that $\vdash_L \ulcorner\psi\urcorner \to \ulcorner\chi\urcorner$. Thus by the inductive hypothesis we have that

$$\vdash_L \varphi_1(\ulcorner\psi\urcorner) \to \varphi_1(\ulcorner\chi\urcorner)$$
$$\vdash_L \varphi_2(\ulcorner\psi\urcorner) \to \varphi_2(\ulcorner\chi\urcorner).$$

Case $\varphi = \varphi_1 \wedge \varphi_2$.

By the propositional calculus we immediately get that

$$\vdash_L \varphi_1 \wedge \varphi_2(\ulcorner\psi\urcorner) \to \varphi_1 \wedge \varphi_2(\ulcorner\chi\urcorner).$$

Case $\varphi = \varphi_1 \vee \varphi_2$.

Also direct from the propositional calculus.

Case $\varphi = \Box_i\varphi_1$.

By applying necessitation to the L-theorem above we get that

$$\vdash_L \Box_i \left(\varphi_1 \left(\ulcorner \psi \urcorner\right) \to \varphi_1 \left(\ulcorner \chi \urcorner\right)\right)$$

and then by the K axiom we get that

$$\vdash_L \Box_i\varphi_1 \left(\ulcorner \psi \urcorner\right) \to \Box_i\varphi_1 \left(\ulcorner \chi \urcorner\right).$$

Case $\varphi = \Diamond_i\varphi_1$.

Follows in a similar way except we use the contrapositive of the L-theorem above and this version of the K axiom:

$$\Box_i(\neg q \to \neg p) \to (\Diamond_i p \to \Diamond_i q).$$

□

Positive formulae behave nicely with respect to ultrafilters in one particular sense:

Proposition 7.27. *Let* $\vec{\mathcal{U}} = \langle \mathcal{U}_\alpha \mid \alpha \in G\rangle$ *be a sequence of ultrafilters on* Ω *and let* $\varphi \in \mathcal{S}(Q)$ *be a positive formula. Then for each* $\alpha_0 \in G$*:*

$$\|\varphi\|_{\bigotimes \vec{\mathcal{U}}} \subseteq \|\varphi\|_{\mathcal{U}_{\alpha_0}}.$$

Proof. Let $\alpha_0 \in G$ and suppose that $x \in \neg\|\varphi\|_{\mathcal{U}_{\alpha_0}}$. For $\alpha \in G$ define

$$U_\alpha := \begin{cases} \{\ulcorner \mu \urcorner \mid \varphi\left(\ulcorner \mu \urcorner\right) \notin x\} \in \mathcal{U}_\alpha & \text{if } \alpha = \alpha_0 \\ \Omega \in \mathcal{U}_\alpha & \text{otherwise.} \end{cases}$$

Let $\Delta = \Delta\left(\{\alpha_0\}\right) \in \Gamma(G)$ and so $\bigotimes_\Delta \vec{U} \in \bigotimes \vec{\mathcal{U}}$. So when we have shown the following we will have demonstrated that $x \notin \|\varphi\|_{\bigotimes \vec{\mathcal{U}}}$:

$\underline{\left(\forall \ulcorner \mu \urcorner \in \bigotimes_\Delta \vec{U}\right) \left[\varphi\left(\ulcorner \mu \urcorner\right) \notin x\right].}$

Let $\ulcorner \mu \urcorner \in \bigotimes_\Delta \vec{U}$. Then $\ulcorner \mu \urcorner = \ulcorner \bigwedge_{\alpha \in F} \mu_\alpha \urcorner$ for some $F \in \Delta$ and

$$(\forall \alpha \in F) \left[\ulcorner \mu_\alpha \urcorner \in U_\alpha\right].$$

Thus $\vdash_L \ulcorner \mu \urcorner \to \ulcorner \mu_{\alpha_0} \urcorner$ and thus by monotonicity (Proposition 7.26) $\vdash_L \varphi\left(\ulcorner \mu \urcorner\right) \to \varphi\left(\ulcorner \mu_{\alpha_0} \urcorner\right)$. But $\varphi\left(\ulcorner \mu_{\alpha_0} \urcorner\right) \notin x$ since $\ulcorner \mu_{\alpha_0} \urcorner \in U_{\alpha_0}$. So $\varphi\left(\ulcorner \mu \urcorner\right) \notin x$ as required.

□

We cannot, in general, say much about the reverse inclusion, however we are able to guarantee it if we are dealing with a much smaller class of formulae:

Definition 7.28. We call a positive formula $\varphi \in \mathcal{S}(Q)$ *super-strongly-positive*[3] iff

$$\left(\forall \ulcorner\psi\urcorner, \ulcorner\chi\urcorner \in \ulcorner\mathcal{S}(P)\urcorner\right) \left[\vdash_L \varphi\left(\ulcorner\psi \wedge \chi\urcorner\right) \leftrightarrow \varphi\left(\ulcorner\psi\urcorner\right) \wedge \varphi\left(\ulcorner\chi\urcorner\right)\right].$$

Examples of such formulae abound, e.g., $\Box_i{}^n(d(Z_0) \vee \Diamond_j \top)$ for any $n \in \omega$, $i, j \in Idx$.

Proposition 7.29. *Let $\varphi \in \mathcal{S}(Q)$ be super-strongly-positive and let $\vec{\mathcal{U}} = \langle \mathcal{U}_\alpha \mid \alpha \in G\rangle$ be a sequence of ultrafilters over Ω. Then*

$$\|\varphi\|_{\bigotimes \vec{\mathcal{U}}} = \bigcap_{\alpha \in G} \|\varphi\|_{\mathcal{U}_\alpha}.$$

Proof. We have the inclusion ($\subseteq$) by the previous proposition.

Suppose now that $x \in \|\varphi\|_{\mathcal{U}_\alpha}$ for each $\alpha \in G$. Set $U_\alpha := \{\ulcorner\mu\urcorner \mid \varphi\left(\ulcorner\mu\urcorner\right) \in x\} \in \mathcal{U}_\alpha$. We will show that $x \in \|\varphi\|_{\bigotimes \vec{\mathcal{U}}}$ by showing that:
$\left(\forall \ulcorner\mu\urcorner \in \bigotimes_{\Delta(\varnothing)} \vec{U}\right) \left[\varphi\left(\ulcorner\mu\urcorner\right) \in x\right]$

Let $\ulcorner\mu\urcorner \in \bigotimes_{\Delta(\varnothing)} \vec{U}$. Thus there exists a finite $F \subseteq G$ such that $\ulcorner\mu\urcorner = \ulcorner\bigwedge_{\alpha \in F} \mu_\alpha\urcorner$ and $(\forall \alpha \in F)\left[\ulcorner\mu_\alpha\urcorner\right] \in U_\alpha$. By $\varphi \in \mathcal{S}(Q)$ being super-strongly-positive we get that

$$\vdash_L \varphi\left(\ulcorner\mu\urcorner\right) \leftrightarrow \bigwedge_{\alpha \in F} \varphi\left(\ulcorner\mu_\alpha\urcorner\right).$$

By $\ulcorner\mu_\alpha\urcorner \in U_\alpha$ we have that $\varphi\left(\ulcorner\mu_\alpha\urcorner\right) \in x$. Thus $\bigwedge_{\alpha \in F} \varphi\left(\ulcorner\mu_\alpha\urcorner\right) \in x$ and so $\varphi\left(\ulcorner\mu\urcorner\right) \in x$.

□

7.7 Positive Formulae and the Separation of Closed Sets

We will now look at the similarities between positive formulae and intensional operators and then look at the similarities between super-strongly-positive for-

[3] Unlike most of our definitions, this definition is not based on the structure of the formula, but rather on the underlying logic itself. A patient reader will see in Section 7.9 that this definition, and the results that follow from it, allow us to conclude canonicity for a very large class of logics.

mulae and the $\Box_i$ operator. First, however, we need to discuss closed sets in $\ulcorner X \urcorner$.

Definition 7.30. For $\ulcorner Z \urcorner \subseteq \ulcorner X \urcorner$ we define $V(\ulcorner Z \urcorner)$ to be the set of pairs $\langle \ulcorner C \urcorner, \ulcorner K \urcorner \rangle$ such that $\ulcorner C \urcorner$ and $\ulcorner K \urcorner$ are closed and

$$\ulcorner C \urcorner \subseteq \ulcorner Z \urcorner \subseteq \neg \ulcorner K \urcorner.$$

We call such a pair $\ulcorner Z \urcorner$*-valid*. We say that the basic-open set $\|\ulcorner \psi \urcorner\|$ separates the pair $\langle \ulcorner C \urcorner, \ulcorner K \urcorner \rangle$ iff $\ulcorner C \urcorner \subseteq \|\ulcorner \psi \urcorner\| \subseteq \neg \ulcorner K \urcorner$. In this case we write $\langle \ulcorner C \urcorner | \ulcorner \psi \urcorner | \ulcorner K \urcorner \rangle$.

Lemma 7.31. *Every* $\ulcorner Z \urcorner$*-valid pair can be separated.*

Proof. Let $\langle \ulcorner C \urcorner, \ulcorner K \urcorner \rangle$ be a $\ulcorner Z \urcorner$-valid pair and let $Y \in \mathcal{P}(X)$. Thus $C(Y) \subseteq Z(Y) \subseteq \neg K(Y)$ and so $C(Y) \cap K(Y) = \varnothing$ and both are closed sets in a compact space. Hence there exists a $\psi(Y) \in \mathcal{S}(P)$ such that $C(Y) \subseteq \|\psi(Y)\| \subseteq \neg K(Y)$. Taking $\ulcorner \psi \urcorner$ to be so defined we get that $\langle \ulcorner C \urcorner | \ulcorner \psi \urcorner | \ulcorner K \urcorner \rangle$. $\Box$

The $\|\ulcorner \psi \urcorner\|$ so found can be considered to be a crude "approximation" to $\ulcorner Z \urcorner$, at least when we compare Z to the components of $\langle \ulcorner C \urcorner, \ulcorner K \urcorner \rangle$.

We will now show that these "approximations" can always be found to satisfy reasonable conditions with respect to positive formulae.

Recall the definition of R_i where we have that $x \notin \check{I}_i(Y)$ means that

$$(\forall y \in Y)\,[\langle x, y \rangle \notin R_i]$$

or less poetically $(\forall y \in Y)(\exists \psi \in y)[\Diamond_i \psi \notin x]$. The $\{x\}$ and $\{y\}$ can be thought of as singleton sets which really are closed sets, and so we have an excellent precedent for the next theorem.

Theorem 7.32. *Let* $\varphi \in \mathcal{S}(Q)$ *be a positive formula, let* $x \in X$ *and suppose that* $x \notin \Phi(\ulcorner Z \urcorner)$. *Then*

$$(\forall \langle \ulcorner C \urcorner, \ulcorner K \urcorner \rangle \in V(\ulcorner Z \urcorner))\left(\exists \ulcorner \psi \urcorner \in \ulcorner \mathcal{S}(P) \urcorner\right)[\varphi(\ulcorner \psi \urcorner) \notin x \text{ and } \langle \ulcorner C \urcorner | \ulcorner \psi \urcorner | \ulcorner K \urcorner \rangle].$$

Proof. By induction on the complexity of φ

Base Cases: If φ is one of the constants $\top$ or $\bot$ there is nothing to do as the result holds by the previous lemma. So suppose that $\varphi = d(Y_0)$.

Let $\langle \ulcorner C \urcorner, \ulcorner K \urcorner \rangle \in V(\ulcorner Z \urcorner)$ and let $x \notin Y_0 = \Phi(\ulcorner D \urcorner)$. Define $\ulcorner K' \urcorner$ as follows:

$$K'(Y) = \begin{cases} K(Y) \cup \{x\} & \text{if } Y = Y_0 \\ K(Y) & \text{otherwise.} \end{cases}$$

We have that $\ulcorner K' \urcorner$ is the union of two closed sets and is thus closed, moreover $\ulcorner C \urcorner \subseteq \ulcorner Z \urcorner \subseteq \ulcorner K' \urcorner$. Thus we can apply the previous lemma to get the desired result for if $\left\langle \ulcorner C \urcorner | \ulcorner \psi \urcorner | \ulcorner K' \urcorner \right\rangle$ then

$$x \notin K'(Y_0) \supset \|\psi(Y_0)\| = \|\varphi(\ulcorner \psi \urcorner)\| .$$

Inductive Hypothesis: Assume that the result holds for φ_0 and φ_1.
Inductive Step: Say that φ is built up from φ_0 and φ_1 in one step.

Fix $\left\langle \ulcorner C \urcorner, \ulcorner K \urcorner \right\rangle \in V(\ulcorner Z \urcorner)$. For each $i < 2$ and $x \in \neg\Phi_i(\ulcorner Z \urcorner)$ let $\ulcorner \psi_i(x) \urcorner \in \ulcorner \mathcal{S}(Q) \urcorner$ be given by the inductive hypothesis, i.e., $\varphi_i\left(\ulcorner \psi_i(x) \urcorner\right) \notin x$ and $\left\langle \ulcorner C \urcorner | \ulcorner \psi_i(x) \urcorner | \ulcorner K \urcorner \right\rangle$.

Suppose now that $x \notin \Phi(\ulcorner Z \urcorner)$.

Case $\varphi = \varphi_0 \vee \varphi_1$.

Thus $x \notin \Phi_0(\ulcorner Z \urcorner) \cup \Phi_1(\ulcorner Z \urcorner)$ and so $x \notin \Phi_0(\ulcorner Z \urcorner)$ and $x \notin \Phi_1(\ulcorner Z \urcorner)$. Set $\ulcorner \psi \urcorner = \ulcorner \psi_0(x) \wedge \psi_1(x) \urcorner$. We then have that

$$\ulcorner C \urcorner \subseteq \left\| \ulcorner \psi_0(x) \urcorner \right\| \cap \left\| \ulcorner \psi_1(x) \urcorner \right\| = \|\ulcorner \psi \urcorner\| \subseteq \left\| \ulcorner \psi_0(x) \urcorner \right\| \subseteq \neg \ulcorner K \urcorner,$$

so $\left\langle \ulcorner C \urcorner | \ulcorner \psi \urcorner | \ulcorner K \urcorner \right\rangle$.

Also,

$$(\forall j < 2)\left[\vdash_L \varphi_j(\ulcorner \psi \urcorner) \rightarrow \varphi_j\left(\ulcorner \psi_j(x) \urcorner\right)\right]$$

and so $(\forall j < 2)\,[\varphi_j(\ulcorner \psi \urcorner) \notin x]$ giving $\varphi(\ulcorner \psi \urcorner) = (\varphi_0 \vee \varphi_1)(\ulcorner \psi \urcorner) \notin x$ as desired.

Case $\varphi = \varphi_0 \wedge \varphi_1$.

Thus $x \notin \varphi_0(\ulcorner Z \urcorner) \cap \varphi_1(\ulcorner Z \urcorner)$, so for some $j < 2$, $x \notin \varphi_j(\ulcorner Z \urcorner)$. Put $\ulcorner \psi \urcorner = \ulcorner \psi_j(x) \urcorner$, and so $\left\langle \ulcorner C \urcorner | \ulcorner \psi \urcorner | \ulcorner K \urcorner \right\rangle$ and we have that $\varphi_j(\ulcorner \psi \urcorner) \notin x$. Thus $\varphi_j(\ulcorner \psi \urcorner) \wedge \varphi_{1-j}(\ulcorner \psi \urcorner) \notin x$, i.e., $\varphi(\ulcorner \psi \urcorner) \notin x$.

Case $\varphi = \Box_i \varphi_1$.

Thus $x \notin I_i(\Phi_1(\ulcorner Z \urcorner))$. So there exists a $y \in R_i(x)$ such that $y \notin \Phi_1(\ulcorner Z \urcorner)$. Put $\ulcorner \psi \urcorner = \ulcorner \psi_1(y) \urcorner$ so we have $\left\langle \ulcorner C \urcorner | \ulcorner \psi \urcorner | \ulcorner K \urcorner \right\rangle$ and $\varphi_1(\ulcorner \psi \urcorner) \notin y$. By the definition of the relation R_i we then get that $\varphi(\ulcorner \psi \urcorner) = \Box_i \varphi_1(\ulcorner \psi \urcorner) \notin x$.

Case $\varphi = \Diamond_i \varphi_1$.

Thus $x \notin \breve{I}_i(\Phi_1(\ulcorner Z \urcorner))$. So for each $y \in R_i(x)$, $y \notin \Phi_1(\ulcorner Z \urcorner)$, i.e., $R_i(x) \subseteq \neg\Phi_1(\ulcorner Z \urcorner)$. Now, for all $y \in R(x)$ we have $\varphi_1\left(\ulcorner \psi_1(y) \urcorner\right)$ and

$\langle \ulcorner C \urcorner | \ulcorner \psi \urcorner | \ulcorner K \urcorner \rangle$. Thus for all $y \in R_i(x)$, $y \in \left\| \neg\varphi_1 \left(\ulcorner \psi_1(y) \urcorner \right) \right\|$, so

$$\left\{ \left\| \neg\varphi_1 \left(\ulcorner \psi_1(y) \urcorner \right) \right\| \mid y \in R_i(x) \right\}$$

is an open cover of $R_i(x)$ a compact set. So we must have an $m \in \omega$ and $y_j \in R_i(x)$ for $j < m$ such that

$$R_i(x) \subseteq \bigcup_{j<m} \left\| \neg\varphi_1 \left(\ulcorner \psi_1(y_j) \urcorner \right) \right\| .$$

Now set $\ulcorner \psi \urcorner = \ulcorner \bigwedge_{j<m} \psi_1(y_j) \urcorner$. Note that we still have $\ulcorner C \urcorner \subseteq \| \ulcorner \psi \urcorner \| \subseteq \neg \ulcorner K \urcorner$ so $\langle \ulcorner C \urcorner | \ulcorner \psi \urcorner | \ulcorner K \urcorner \rangle$ still holds. We also have by φ_1 being positive that

$$(\forall j < m) \left[\vdash_L \varphi_1 \left(\ulcorner \psi \urcorner \right) \to \varphi_1 \left(\ulcorner \psi_1(y_j) \urcorner \right) \right]$$

which tells us that

$$(\forall j < m) \left[\left\| \neg\varphi_1 \left(\ulcorner \psi_1(y_j) \urcorner \right) \right\| \subseteq \left\| \neg\varphi_1 \left(\ulcorner \psi \urcorner \right) \right\| \right] .$$

Thus $R_i(x) \subseteq \| \neg\varphi_1 (\ulcorner \psi \urcorner) \|$, so $x \in I_i (\| \neg\varphi_1 (\ulcorner \psi \urcorner) \|) = \| \Box_i \neg\varphi_1 (\ulcorner \psi \urcorner) \|$ telling us that $\Box_i \neg\varphi_1 (\ulcorner \psi \urcorner) \in x$. Hence $\varphi (\ulcorner \psi \urcorner) = \Diamond_i \varphi_1 (\ulcorner \psi \urcorner) \notin x$.

□

The class of super-strongly-positive formulae are essentially □-like operators as we can now make clear.

Definition 7.33. For $\varphi \in \mathcal{S}(Q)$ a super-strongly-positive formula let

$$\ulcorner R_\varphi(x) \urcorner = \bigcap \left\{ \| \ulcorner \psi \urcorner \| \mid \varphi (\ulcorner \psi \urcorner) \in x \right\} .$$

Note that this is a pointwise intersection and hence is a closed set.

Lemma 7.34. *For $\varphi \in \mathcal{S}(Q)$ a super-strongly-positive formula and $\ulcorner Z \urcorner \in \ulcorner \mathcal{P}(X) \urcorner$*

$$\ulcorner R_\varphi(x) \urcorner \subseteq \ulcorner Z \urcorner \Longleftrightarrow x \in \Phi (\ulcorner Z \urcorner).$$

Proof. We prove the if and only if parts separately.

($\Longleftarrow$) Suppose that $x \in \Phi (\ulcorner Z \urcorner)$ and *not* $\left(\ulcorner R_\varphi(x) \urcorner \subseteq \ulcorner Z \urcorner \right)$. Thus there is a closed $\ulcorner C \urcorner \subseteq \ulcorner R_\varphi(x) \urcorner - \ulcorner Z \urcorner \subseteq \neg \ulcorner Z \urcorner$ with some $C(Y_0) \neq \varnothing$. So $\langle \ulcorner C \urcorner , \ulcorner \varnothing \urcorner \rangle \in V (\neg \ulcorner Z \urcorner)$. Also $x \in \neg \check{\Phi} (\neg \ulcorner Z \urcorner)$ and $\check{\varphi}$ is positive. So by Theorem 7.32 there is a $\ulcorner \psi \urcorner$ such that $\langle \ulcorner C \urcorner | \ulcorner \psi \urcorner | \ulcorner \varnothing \urcorner \rangle$ and $\check{\varphi} (\ulcorner \psi \urcorner) \notin x$.

Thus $\varphi\left(\ulcorner\neg\psi\urcorner\right) \in x$ and $\ulcorner C\urcorner \subseteq \|\ulcorner\psi\urcorner\|$. Hence $\ulcorner R_\varphi(x)\urcorner \subseteq \|\ulcorner\neg\psi\urcorner\| \subseteq \neg\ulcorner C\urcorner$ telling us that $\ulcorner C\urcorner \subseteq \neg\ulcorner R_\varphi(x)\urcorner \cap \ulcorner R_\varphi(x)\urcorner = \varnothing$ contradicting $C(Y_0) \neq \varnothing$.

($\Longrightarrow$) Suppose that $\ulcorner R_\varphi(x)\urcorner \subseteq \ulcorner Z\urcorner$ and $x \notin \Phi\left(\ulcorner Z\urcorner\right)$. Since $\left\langle \ulcorner R_\varphi(x)\urcorner, \ulcorner\varnothing\urcorner \right\rangle \in V\left(\ulcorner Z\urcorner\right)$ we have that there is a $\ulcorner\psi\urcorner \in \ulcorner\mathcal{S}(P)\urcorner$ separating the pair and which satisfies

$$\bigcap\{\|\ulcorner\chi\urcorner\| \mid \varphi\left(\ulcorner\chi\urcorner\right) \in x\} = \ulcorner R_\varphi(x)\urcorner \text{ and } \varphi\left(\ulcorner\psi\urcorner\right) \notin x.$$

Since φ only mentions finitely many variables, without loss of generality we can assume that $\psi(Y) = \top$ for all but finitely many Y. Thus by compactness there exists an $m \in \omega$ and $\ulcorner\chi_j\urcorner \in \ulcorner\mathcal{S}(P)\urcorner$ such that

$$(\forall j < m)\left[\varphi\left(\ulcorner\chi_j\urcorner\right) \in x\right] \text{ and } \bigcap_{j<m} \|\ulcorner\chi_j\urcorner\| \subseteq \|\ulcorner\psi\urcorner\|.$$

We then put $\ulcorner\chi\urcorner = \ulcorner\bigwedge_{j<m}\chi_j\urcorner$ and we have $\varphi\left(\ulcorner\chi\urcorner\right) \in x$ by φ being super-strongly-positive and $\|\ulcorner\chi\urcorner\| \subseteq \|\ulcorner\psi\urcorner\|$, i.e., $\vdash_L \ulcorner\chi\urcorner \to \ulcorner\psi\urcorner$. Thus by φ being a positive formula we have that $\varphi\left(\ulcorner\psi\urcorner\right) \in x$, a contradiction.

□

7.8 Finding the Right Ultrafilters for Sahlqvist's Theorem

We can now put our results together to find an ultrafilter which will allow us to get the Sahlqvist result. In particular we will find an ultrafilter $\mathcal{U}$ which will satisfy the following conditions:

1. $(\forall Z \in \mathcal{P}(X))\,[\|d(Z)\|_{\mathcal{U}} = Z]$,
2. $(\forall\varphi \in \mathcal{S}(Q)$ super-strongly-positive$)\,\left[\|\varphi\|_{\mathcal{U}} = \Phi\left(\ulcorner D\urcorner\right)\right]$, and
3. $(\forall\varphi \in \mathcal{S}(Q)$ positive$)\,\left[\|\varphi\|_{\mathcal{U}} \subseteq \Phi\left(\ulcorner D\urcorner\right)\right]$

These conditions will allow us to:

1. Show that each set is expressible,
2. Deal with the antecedent part of a Sahlqvist formula, and
3. Quickly dispatch the consequent part.

Fix, for the moment an $x \in X$ and a positive $\varphi \in \mathcal{S}(Q)$.

Definition 7.35. For $\langle \ulcorner C \urcorner, \ulcorner K \urcorner \rangle \in V(\ulcorner D \urcorner)$ let

$$U(\ulcorner C \urcorner, \ulcorner K \urcorner) = \{\ulcorner \mu \urcorner \in \Omega \mid \langle \ulcorner C \urcorner | \ulcorner \mu \urcorner | \ulcorner K \urcorner \rangle \text{ and } (x \notin \Phi(\ulcorner D \urcorner) \Longrightarrow \varphi(\ulcorner \mu \urcorner) \notin x)\}.$$

Lemma 7.36. $\Theta := \{U(\ulcorner C \urcorner, \ulcorner K \urcorner) \mid \langle \ulcorner C \urcorner, \ulcorner K \urcorner \rangle \in V(\ulcorner D \urcorner)\}$ *has the finite intersection property.*

Proof. Theorem 7.32 immediately guarantees to us that $\varnothing \notin \Theta$ and if

$$\langle \ulcorner C_0 \urcorner, \ulcorner K_0 \urcorner \rangle, \langle \ulcorner C_1 \urcorner, \ulcorner K_1 \urcorner \rangle \in V(\ulcorner D \urcorner)$$

then

$$\ulcorner C_0 \urcorner \cup \ulcorner C_1 \urcorner \subseteq \ulcorner Z \urcorner \subseteq \neg \ulcorner K_0 \urcorner \cap \neg \ulcorner K_1 \urcorner$$

so $\langle \ulcorner C_0 \urcorner \cup \ulcorner C_1 \urcorner, \ulcorner K_0 \urcorner \cup \ulcorner K_1 \urcorner \rangle \in V(\ulcorner D \urcorner)$. Moreover, it is clear that if

$$\langle \ulcorner C_0 \urcorner \cup \ulcorner C_1 \urcorner | \ulcorner \mu \urcorner | \ulcorner K_0 \urcorner \cup \ulcorner K_1 \urcorner \rangle$$

then for $i < 2$ $\langle \ulcorner C \urcorner_i | \ulcorner \mu \urcorner | \ulcorner K \urcorner_i \rangle$. This establishes that $U(\ulcorner C_0 \urcorner, \ulcorner K_0 \urcorner) \cap U(\ulcorner C_1 \urcorner, \ulcorner K_1 \urcorner) \supseteq U(\ulcorner C_0 \urcorner \cup \ulcorner C_1 \urcorner, \ulcorner K_0 \urcorner \cup \ulcorner K_1 \urcorner)$. Notice how there is nothing to show here for the second condition on membership in $U(\ulcorner C \urcorner, \ulcorner K \urcorner)$. □

Choose $\mathcal{U}$ to be an ultrafilter which extends Θ.

Lemma 7.37. *For all* $\psi \in \mathcal{S}(Q)$ *super-strongly-positive*

$$\|\psi\|_{\mathcal{U}} \supseteq \Psi(\ulcorner D \urcorner).$$

Proof. Let $y \in \Psi(\ulcorner D \urcorner)$. Thus by Lemma 7.34 $\ulcorner R_\psi(y) \urcorner \subseteq \ulcorner D \urcorner$, telling us that

$$\langle \ulcorner R_\psi(y) \urcorner, \ulcorner \varnothing \urcorner \rangle \in V(\ulcorner D \urcorner).$$

$$\underline{\left(\forall \ulcorner \mu \urcorner \in U(\ulcorner R_\psi(y) \urcorner, \ulcorner \varnothing \urcorner)\right) [\psi(\ulcorner \mu \urcorner) \in y]}$$

Let $\ulcorner \mu \urcorner \in U(\ulcorner R_\psi(y) \urcorner, \ulcorner \varnothing \urcorner)$. Thus $\ulcorner R_\psi(y) \urcorner \subseteq \|\ulcorner \mu \urcorner\|$ and Lemma 7.34 allows us to conclude that $y \in \Psi(\|\ulcorner \mu \urcorner\|) = \|\psi(\ulcorner \mu \urcorner)\|$, so $\psi(\ulcorner \mu \urcorner) \in y$.

This establishes that $y \in \|\psi\|_{\mathcal{U}}$. □

Remember that x and φ are fixed for the moment.

Lemma 7.38. $x \in \|\varphi\|_{\mathcal{U}} \Longrightarrow x \in \Phi(\ulcorner D \urcorner)$.

Proof. Suppose that $x \notin \Phi\left(\ulcorner D \urcorner\right)$
$\underline{(\forall \ulcorner \mu \urcorner \in U(\ulcorner \varnothing \urcorner, \ulcorner \varnothing \urcorner))\ [\varphi\left(\ulcorner \mu \urcorner\right) \notin x]}$

Let $\ulcorner \mu \urcorner \in U(\ulcorner \varnothing \urcorner, \ulcorner \varnothing \urcorner)$. Since $x \notin \Phi\left(\ulcorner D \urcorner\right)$ we must have that $\varphi\left(\ulcorner \mu \urcorner\right) \notin x$. This establishes that $x \notin \|\varphi\|_{\mathcal{U}}$. □

Now that we have these results in hand let $\mathcal{U}_{(x,\varphi)}$ be the name for the ultrafilter so resulting from each choice of x and φ. Also, we make the following two definitions:

$$\vec{\mathcal{U}} = \langle \mathcal{U}_{(x,\varphi)} \mid x \in X, \varphi \in \mathcal{S}(Q) \text{ is positive} \rangle, \text{ and}$$
$$\mathcal{U} = \bigotimes \vec{\mathcal{U}}.$$

Lemma 7.39. *The ultrafilter $\mathcal{U}$ satisfies conditions (1), (2), and (3) given at the start of this section.*

Proof. Notice that (1) is just a special case of (2). We prove (3) first:
For (3): Let $x \in X$, $\varphi \in \mathcal{S}(Q)$ be positive. Suppose that $x \in \|\varphi\|_{\mathcal{U}} \subseteq \|\varphi\|_{\mathcal{U}_{(x,\varphi)}}$ by Proposition 7.27. Thus $x \in \Phi\left(\ulcorner D \urcorner\right)$ by Lemma 7.38.
For (2): Let $\psi \in \mathcal{S}(Q)$ be super-strongly-positive. Thus $\|\psi\|_{\mathcal{U}} \subseteq \Psi\left(\ulcorner D \urcorner\right)$ because ψ is positive and we already have (3).
We have that $\|\psi\|_{\mathcal{U}_{(x,\varphi)}} \supseteq \Psi\left(\ulcorner D \urcorner\right)$ by Lemma 7.37. Thus

$$\Psi\left(\ulcorner D \urcorner\right) \supseteq \|\psi\|_{\mathcal{U}} \supseteq \bigcap_{(x,\varphi)} \|\psi\|_{\mathcal{U}_{(x,\varphi)}} \supseteq \Psi\left(\ulcorner D \urcorner\right).$$

□

Now we can attack the essence of Sahlqvist's Theorem:

Definition 7.40. A *sub-Sahlqvist* formula is one of the form

$$\psi\left(\ulcorner \varphi \urcorner\right) \to \chi,$$

where

1. ψ is dual-strongly-positive,
2. $\ulcorner \varphi \urcorner$ is a sequence of formulae which are either
   (a) super-strongly positive, or
   (b) the negation of a positive formula, and
3. χ is a positive formula.

Lemma 7.41. *If* $\theta = \psi\left(\ulcorner\varphi\urcorner\right) \to \xi \in \mathcal{S}(Q)$ *is sub-Sahlqvist then it satisfies:*

$$\|\psi\left(\ulcorner\varphi\urcorner\right) \to \xi\|_{\mathcal{U}} \subseteq \neg\Psi\left(\ulcorner\Phi\urcorner\left(\ulcorner D\urcorner\right)\right) \cup \Xi\left(\ulcorner D\urcorner\right),$$

i.e., $\|\theta\|_{\mathcal{U}} \subseteq \Theta\left(\ulcorner D\urcorner\right)$.

Proof. We start with the following claim:
$\ulcorner\Phi\urcorner\left(\ulcorner D\urcorner\right) \subseteq \|\ulcorner\varphi\urcorner\|_{\mathcal{U}}$.

Suppose that φ is a formula which satisfies one of the sub-requirements (a) or (b) in requirement 2 of Definition 7.40. So we have 2 cases:

Case φ is super-strongly positive.
Thus $\Phi\left(\ulcorner D\urcorner\right) = \|\varphi\|_{\mathcal{U}}$ by condition (2).

Case $\varphi = \neg\theta$, θ a positive formula.
Thus $\|\theta\|_{\mathcal{U}} \subseteq \Theta\left(\ulcorner D\urcorner\right)$ by condition (3). Hence

$$\Phi\left(\ulcorner D\urcorner\right) = \neg\Theta\left(\ulcorner D\urcorner\right) \subseteq \neg\|\theta\|_{\mathcal{U}} = \|\neg\theta\|_{\mathcal{U}} = \|\varphi\|_{\mathcal{U}}.$$

By ψ being positive, we see that Ψ is monotonic (Proposition 7.26 (2)) and together with Proposition 7.25 we get that

$$\Psi\left(\ulcorner\Phi\urcorner\left(\ulcorner D\urcorner\right)\right) \subseteq \Psi\left(\|\ulcorner\varphi\urcorner\|_{\mathcal{U}}\right) \subseteq \|\psi\left(\ulcorner\varphi\urcorner\right)\|_{\mathcal{U}}.$$

Now let $x \in \|\psi\left(\ulcorner\varphi\urcorner\right) \to \xi\|_{\mathcal{U}}$ and suppose that

$$x \in \Psi\left(\ulcorner\Phi\urcorner\left(\ulcorner D\urcorner\right)\right) \subseteq \|\psi\left(\ulcorner\varphi\urcorner\right)\|_{\mathcal{U}}.$$

Thus $x \in \|\xi\|_{\mathcal{U}} \subseteq \Xi\left(\ulcorner D\urcorner\right)$. □

Definition 7.42. We say a formula is Sahlqvist iff it is of the form $\psi\left(\ulcorner\varphi\urcorner\right)$ for ψ strongly positive and each $\varphi(Z)$ sub-Sahlqvist.

Theorem 7.43. *Suppose that* L *contains a Sahlqvist formula* ξ. *Then* $(X, R) \vDash \xi$.

Proof. Let v be a valuation on the frame (X, R). Rename the propositional letters of ξ so that it can be of the form $\psi\left(\ulcorner\varphi\urcorner\right)$ for ψ strongly-positive, each $\varphi(Z)$ is sub-Sahlqvist *and* each propositional letter $p \in \mathbb{P}$ renamed so that it becomes $d\left(v\left(p\right)\right)$. Thus

$$\begin{aligned} X &= \|\psi\left(\ulcorner\varphi\urcorner\right)\|_{\mathcal{U}} \\ &\subseteq \Psi\left(\|\ulcorner\varphi\urcorner\|_{\mathcal{U}}\right) \\ &\subseteq \Psi\left(\ulcorner\Phi\urcorner\left(\ulcorner D\urcorner\right)\right) \end{aligned}$$

$$= \Xi\left(\ulcorner D \urcorner\right),$$

where the first line is by Proposition 7.8, the second by Proposition 7.23 and the third by Lemma 7.41 and Proposition 7.26. Since $\ulcorner D \urcorner$ agrees with the valuation we have that $(X, R, v) \models \xi$. □

7.9 A Comparison With Existing Approaches to Sahlqvist Logic

The earliest approach to Sahlqvist's Theorem is naturally SAHLQVIST's own result [68] which worked at building the correspondence between modal formulae and first order formulae in a natural manner. This correspondence then gave us, via Fine's Theorem, the desired canonicity result. Sahlqvist's Theorem has been generalised from its original incarnation to take into account multi-modal operators and to maximize the class of axioms which it encompasses. A very revealing example is KRACHT's treatment in [50, Section 5.5] which leads the reader through a wide reaching calculus of correspondence that then gives rise to the Sahlqvist result. These results are all upfront about the role played by elementary conditions on the accessibility relation and are, of course, more revealing about the particular logics.

Our idea of doing away with elementary conditions is not unique. Since the author first detailed this work in the technical report [87] he has become aware of a small but significant collection of papers which approach Sahlqvist's theorem from an algebraic point of view. In each of these works it is made explicit that the underlying topology and the closed sets that the topology gives rise to are important, and of equal importance is the way that positive formulae respect these closed sets. In this section we will attempt to outline the alternate approach and indicate how it produces a comparable range of canonicity results, however a full analysis of the relationships between the alternate approach and the one given in the preceding sections is beyond the scope of this chapter.

The first and undeniably foundational paper is BJARNI JÒNSSON and ALFRED TARSKI's *Boolean algebras with operators* [44, 45].[4] This paper predated the work by KRIPKE on the relational semantics for modal logics [54] and so was presented from an algebraicist's point of view. JÒNSSON and TARSKI dealt with boolean algebras augmented by operators. Much like ours, their operators were of arbitrary arity and were allowed to exist in any number, however

[4] An overview of this work is given in HENKIN, MONK and TARSKI's book *Cylindric Algebras I* [38, Chapter 2.7] which, naturally enough, treats the results within the context of Cylindric Algebras.

their fundamental operations were essentially dual to the 'necessity' operators of modal logicians—their operators distributed over join (disjunction). The results they presented were extremely deep, involved, and their relevance to modal logic, while apparent, was not very persuasive and it is perhaps for this reason that the Sahlqvist nature of these results was not recognised. Indeed, JÒNSSON himself lamented [43, p. 473] that the techniques of [44] have suffered "decades of neglect" and that "modal logicians are still at best only superficially acquainted with the paper."

The starting point of [44] is to take a boolean algebra $\underline{A}$ and look at the class of operators J on this boolean algebra that satisfy:

1. For each $\overline{a}, \overline{b} \in A$ such that $a_j = b_j$ for all $j < \text{length}\,(a) = \text{length}\,(b) = \text{arity}\,(J)$, $j \neq k < \text{arity}\,(J)$,
$$J(\overline{a} \vee \overline{b}) = J(\overline{a}) \vee J(\overline{b}).$$
2. If $\overline{a} \in A$ is such that $a_j = \bot$ for some j, then $J(\overline{a}) = \bot$.

These operators, which are called *additive*, are dual to operators more familiar to modal logicians, that is those that distribute over meet ($\wedge$). Nevertheless, by simply dualising the arguments of [44] we obtain results that seem to make more sense to modal logicians, and so we will continue our discussion by reporting a dualised version of the contents of [44].

Thus, their basic non-boolean operator was the *normal, multiplicative* operator I which satisfied:

1. For each $\overline{a}, \overline{b} \in A$ such that $a_j = b_j$ for all $j < \text{length}\,(a) = \text{length}\,(b) = \text{arity}\,(I)$, $j \neq k < \text{arity}\,(I)$,
$$I(\overline{a} \wedge \overline{b}) = I(\overline{a}) \wedge I(\overline{b}).$$
2. If $\overline{a} \in A$ is such that $a_j = \top$ for some j, then $I(\overline{a}) = \top$.

They then discussed a specialised type of multiplicative normal operator which allowed for infinite meets in one coordinate. This type of operator was called completely multiplicative.

JÒNSSON and TARSKI then introduced the notion of a perfect extension of a boolean algebra $\underline{A}$ which is essentially[5] the boolean algebra $\underline{B}$ formed from the stone space of $\underline{A}$. They went on to talk about the perfect extension

[5] JÒNSSON and TARSKI actually talked about an arbitrary perfect extension of $\underline{A}$, however they proved that all perfect extensions are isomorphic.

of a multiplicative normal modal operator I by defining it to be the map I^* : ${}^nB \longrightarrow B$ given by[6]

$$I^*(\overline{Z}) = \bigcap_{\substack{\overline{Y}\supseteq\overline{Z}\\ \overline{Y}\text{ open}}} \bigcup_{\substack{\|\overline{a}\|\subseteq\overline{Y}\\ \overline{a}\in A}} \|I(\overline{a})\|.$$

This turns out to be a completely multiplicative operator and, in the case of uni-adic operators, it turns out to be equivalent to the induced operations on the canonical frame. Thus if we can guarantee that the perfect extension of any algebra $\underline{\underline{A}}$ has certain equational conditions in common with $\underline{\underline{A}}$ we will be sure that the move to a canonical frame will preserve the corresponding logical formulae and, consequently, canonicity for particular intensional logics will obtain.

By looking at how the class of normal multiplicative operations behaved under composition, JÒNSSON and TARSKI were able to show the following:[7]

Theorem 7.44 (Jònsson and Tarski Extension Theorem). *Let $\mathcal{K}$ be an equationally definable class of boolean algebras with normal multiplicative operators. The perfect extension of any element of $\mathcal{K}$ is in $\mathcal{K}$.*

Unfortunately for the cause of canonicity, by an equation JÒNSSON and TARSKI were referring to any algebraic equality whose terms were built up out of normal multiplicative operators, and the boolean operators $\wedge$ and $\vee$. They explicitly did not consider $\neg$. Nevertheless, this result does have immediate and interesting consequences. For instance, consider the 4 axiom: $\Box p \to \Box\Box p$, whose algebraic equivalent is the inequality $I(a) \leqslant I(I(a))$. Turning this into an equation we get

$$I(a) \wedge I(I(a)) = I(a).$$

This is clearly of an appropriate form for the JÒNSSON and TARSKI extension theorem so we instantly get the canonicity of the logic K4. Again, this is without reference to the transitivity of the underlying relation.

The paper [44] did not take its result as far as possible, for consider our rather elementary Theorem 7.10: That

$$\Diamond_i^n p \to \Box_j{}^m p \in L$$

---

[6] Notice how open sets are important in this extension, whereas in our proof of Sahlqvist's Theorem we relied heavily on closed and hence compact sets. This is indicative of a strong dependence on topological notions in the two approaches, and indeed, G. SAMBIN and V. VACCARO [69] have obtained the elementary condition version of Sahlqvist's Theorem through an elegant analysis based on topological notions, including a strong reliance on the closure of $R(x)$.

[7] Theorem 2.18, p. 928 of [44].

is canonical. This formula can not be handled by the JÒNSSON-TARSKI extension theorem because its equational equivalent would be

$$I_i^n(\neg a) \vee I_j^m(a) = \top$$

which contains a forbidden $\neg$. As this seems like such a minor impediment to the application of the extension theorem it is no surprise that [44] was improved upon, with the first result by HUGO RIBEIRO [67]. Specifically, RIBEIRO showed that monotonic, as opposed to multiplicative, functions can be used in certain "closed under composition" lemmata. Later, LEON HENKIN, [37], picked up the trail by extending the extension result to operators which he called 'p-additive', a generalisation of the notion of 'additive' (dual to 'multiplicative'). After a further 24 years, and with the prompting of MAARTEN DE RIJKE and YDE VENEMA's paper [16], which gave a formalised statement of Sahlqvist's Theorem for boolean algebras of arbitrary similarity type, JÒNSSON [43] was then able to gather these results together to show that the whole force of Sahlqvist's Theorem can be derived by following the lead of the original paper [44].[8]

As in [44], and as in our work, JÒNSSON's proof gives canonicity without referring to any elementary conditions on accessibility relations, however, unlike our work so far, JÒNSSON was able to go further and demonstrate that certain non-Sahlqvist logics can be canonical. An example is K4McK which JÒNSSON showed to be canonical by detouring through what he called *quasi-identities*. Translating into the terminology of logical satisfaction these are statements of the form

$$\varphi \Longrightarrow \psi$$

which is true for an algebra $\underline{\underline{A}}$ precisely when

$$\underline{\underline{A}} \models \varphi \Longrightarrow \underline{\underline{A}} \models \psi.$$

Our ultrafilter semantics formalism does not seem to admit such a notion since $\|\varphi\|_u = X$ by itself does not necessarily correspond to a statement about relationships between the R-interior operators and so ultrafilter semantics *appear* to be unable to deal with K4McK canonicity. Nevertheless, K4McK can be dealt with by our Sahlqvist result as follows:

Lemma 7.45. *In the logic* K4McK *the formula* $\varphi := \Box\Diamond p$ *is super-strongly-positive.*

Proof. Work in the canonical *model* of $L =$ K4McK. Since φ is positive and

[8] Another way in which [44] and [37] have been built on is the work [2] which looks at how perfect extensions of an algebra relate to perfect extensions of derived algebras, such as subalgebras and products.

hence monotonic we need only show:
$\Box\Diamond p \wedge \Box\Diamond q \to \Box\Diamond(p \wedge q) \in L$.

Suppose that $\Box\Diamond p, \Box\Diamond q \in x$ in the canonical model for L. To show that $\Box\Diamond(p \wedge q)$ holds at x we show that $\Diamond(p \wedge q)$ holds at all R-successors y. Fix such a y.

By the 4 axiom we have that $\Diamond q$, $\Box\Diamond p$, $\Diamond\Box p$ (by $\Box\Diamond p$ and McK) hold throughout all of $R^*(y)$.[9] Thus there is a $z \in R(y)$ such that $\Box p \in z$ and so by the 4 axiom p holds throughout $R(z)$. Also, there is a $w \in R(z)$ such that $q \in w$. Hence $p \wedge q \in w$, so $\Diamond(p \wedge q) \in y$ by $R(z) \subseteq R(y)$.

$\Box$

Thus, the system K4McK is axiomatised by Sahlqvist formulae and so is canonical.

The determination of exactly how our approach relates to JÒNSSON's for this and similar logics is a quest deserving of further research yet is, unfortunately, outside the scope of this thesis.

The Sahlqvist result given in this chapter does differ slightly from that given by DE RIJKE and VENEMA in [16]. Most obviously there is the difference in the allowable arity of the modal operators. We have only discussed unary normal modal operators however it is reasonable to expect that our approach will generalise. The other difference is that caused by our inclusion of super-strongly-positive in our definition of Sahlqvist formulae. While DE RIJKE and VENEMA talked only about formulae of the form $\Box_{i_0} \cdots \Box_{i_{n-1}} p$, we used the more general notion of a super-strongly-positive formulae of which the $\Box_{i_0} \cdots \Box_{i_{n-1}} p$ formulae are an obvious special case. DE RIJKE-VENEMA Sahlqvist formulae then have the endearing feature of being patently identifiable by their very form whereas we look to a larger class of formulae which may not always be easily identifiable.

This chapter has looked at a marriage of canonicity for non-iterative non-normal logics and canonicity for the normal Sahlqvist logics. Our analysis has highlighted a technique which, while related to those exploited in Chapters 5 and 6, does give us yet another way of looking at canonicity. While JÒNSSON and TARSKI's results may go further in their ultimate conclusions our work does appear new and different and the nature of the true relationship between these two approaches is a topic worthy of further study.

One final comment that arises from the difference between the DE RIJKE-VENEMA notion of "Sahlqvist" and our own: Any logic which has an axioma-

[9] R^* is the reflexive transitive closure of R.

tisation consisting solely of axioms of the form

$$\varphi(\overline{p}) \wedge \varphi(\overline{q}) \to \varphi(\overline{p} \wedge \overline{q}),$$

for φ a positive formula, is canonical since its axioms are almost trivially Sahlqvist—φ is super-strongly-positive. This tells us that some bizarre looking logics are canonical such as ones with axioms drawn from the following list:

$$\Box\Diamond p \wedge \Box\Diamond q \to \Box\Diamond(p \wedge q)$$
$$\Box\Diamond\Box p \wedge \Box\Diamond\Box q \to \Box\Diamond\Box(p \wedge q)$$
$$\Box(p \wedge \Diamond\Box\Diamond q) \wedge \Box(r \wedge \Diamond\Box\Diamond s) \to \Box(p \wedge r \wedge \Diamond\Box\Diamond q \wedge s).$$

These logics do not strike the author as being elementary and so we may wish to consider axioms of this form to see if we can use them as counter-examples to the "canonicity implies elementarity" conjecture.

While we have seen that there is compelling circumstantial evidence to believe that the conjecture is true, we have at last found some indication that it may not be true.[10]

[10] ADDED IN PROOF: The author now suspects that the whole ultrafilter semantics approach will work not only in the canonical frame but also in any ultrapower of the canonical frame. If this is indeed the case then all formulae that can be shown to be canonical via the ultrafilter semantics technique must be elementary. Stay tuned!

Chapter 8

A Non-standard Injection Between Canonical Frames

8.1 Introduction

We now return, full time, to the study of normal modal logics and we will be looking at the problems highlighted in Section 3.7, and in particular we will look at the way to relate canonical frames via our usual repertoire of maps.

In an attempt to make a start on problems to do with the existence of non-standard isomorphisms or automorphisms, we will try to answer the weaker question: Are there any non-standard injective accessibility preserving maps between canonical frames?

In Section 8.3 we develop maps which we call *ultrafilter embeddings*, some of which actually are non-standard, however we won't verify this until Section 8.6 when we will have sufficiently developed the theory of such maps. The particular property of ultrafilter embeddings which we will use is that for any countable collection of points in $F^L_{\omega_1}$ we can find an ultrafilter embedding from F^L_{ω} into $F^L_{\omega_1}$ which will hit every element of that countable collection.

Also in Section 8.6 we will argue that in some instances, an injective accessibility preserving embedding is the only way to relate F^L_{ω} and $F^L_{\omega_1}$ in such a way that we will be guaranteed of hitting some particularly awkward points. We will then conclude this chapter with a discussion in Section 8.7 of how these ultrafilter embeddings actually turn into frame homomorphisms when we deal with logics of bounded alternative. This will set us up for the next chapter where we can use these maps to say something about non-standard isomorphisms and automorphisms.

It is a reasonable hope that the ultrafilter embeddings introduced here will be a basis for further investigation into the canonical frame problems, as well as providing a new tool in the general analysis of canonical frames. The techniques and results of the next chapter will provide some justification for this hope.

8.2 Notation and Basic Definitions

As indicated in Section 2.4 when working within the sphere of normality we will restrict ourselves to normal multi-modal logics of arity one.

The question of this chapter is that of whether there are any interesting maps between canonical frames which are not generated from maps between the underlying language, i.e., maps which are not standard.

Recall the following definition which originally appeared as Definition 2.57.

Definition 8.1. A map $h : X_P \longrightarrow X_Q$ is called *standard* if $h = f_+$ for some homomorphism $f : Q \longrightarrow \mathcal{S}(P)$, and *non-standard* otherwise.

Clearly there are non-standard maps, but are there non-standard homomorphisms? We will not be able to get even an answer to this until at least Section 8.7, and then our solutions will be only partial, however we will be able to answer affirmatively the question of whether there are any non-standard accessibility preserving embeddings of X_P into X_Q for arbitrary logics. The next definition will make this precise.

Definition 8.2. A map $f : (X, \underline{R}) \longrightarrow (Y, \underline{T})$ between frames is an *accessibility preserving embedding* iff

$$(\forall i \in Idx)\,(\forall x_1, x_2 \in X)\,[\langle x_1, x_2\rangle \in R_i \Longleftrightarrow \langle f(x_1), f(x_2)\rangle \in T_i]$$

We use the additional phrase "accessibility preserving" to distinguish this kind of map from the kind described by the plain word "embedding" which, to many, will strongly suggest a frame homomorphic embedding.

8.3 Ultrafilter Embeddings

The basic map which we will use in our construction of a non-standard accessibility preserving embedding is the ultrafilter embedding. The idea behind the construction of these maps is to use an ultrafilter on a collection of standard embeddings to average the output of this collection.[1] Fix, for the moment, $P \subseteq Q$ as sets of propositional variables. Also, set $\Omega_{P,Q} = \{\mu : Q \longrightarrow P \mid \mu\restriction_P = \mathrm{id}\}$ and where unambiguous we will suppress the P, Q so that we just write Ω.

[1] Given a sequence $\langle x_i \rangle_{i \in \Omega}$ in a canonical frame $(X, \underline{R})$, together with an ultrafilter $\mathcal{U}$ on Ω we can define the *ultrafilter average* of $\langle x_i \rangle$ with respect to $\mathcal{U}$ to be the maximal consistent set $\{\varphi \mid \varphi \in x_i\ \mathcal{U}\text{ a.e.}\}$. This construction is not new, as $\langle x_i \rangle$ defines a point within $(X, \underline{R})^{\Omega} / \mathcal{U}$ which can be mapped in a canonical way down onto the 'average' point given above (see, for instance, the map given in the proof of Theorem 11.2.1 in [32]).

Definition 8.3. For $\mathcal{U}$ an ultrafilter on $\Omega_{P,Q}$ define $h_{\mathcal{U}} : X_P \longrightarrow X_Q$ as follows:

$$h_{\mathcal{U}}(z) = \{\varphi \in \mathcal{S}(Q) \mid \{\mu \in \Omega \mid \mu(\varphi) \in z\} \in \mathcal{U}\}.$$

Proposition 8.4. *$h_{\mathcal{U}}$ as given is well defined.*

Proof. We must show that $h_{\mathcal{U}}(z)$ is in fact in X_Q.

$h_{\mathcal{U}}(z)$ is maximal

$$\begin{aligned} \varphi \notin h_{\mathcal{U}}(z) &\Longrightarrow \{\mu \in \Omega \mid \mu(\varphi) \in z\} \notin \mathcal{U} \\ &\Longrightarrow \{\mu \in \Omega \mid \mu(\varphi) \notin z\} \in \mathcal{U} \\ &\Longrightarrow \{\mu \in \Omega \mid \neg\mu(\varphi) \in z\} \in \mathcal{U} \\ &\Longrightarrow \{\mu \in \Omega \mid \mu(\neg\varphi) \in z\} \in \mathcal{U} \\ &\Longrightarrow \neg\varphi \in h_{\mathcal{U}}(z). \end{aligned}$$

$h_{\mathcal{U}}(z)$ is consistent:

Suppose that $\varphi_0, \ldots, \varphi_{n-1} \in h_{\mathcal{U}}(z)$ and assume that $\varphi_0, \ldots, \varphi_{n-1} \vdash_L \bot$. Thus

$$\bigcap_{j<n} \{\mu \in \Omega \mid \mu(\varphi_j) \in z\} \in \mathcal{U}$$

which guarantees us a $\mu \in \Omega$ so that $\mu(\varphi_0), \ldots, \mu(\varphi_{n-1}) \in z$. Now, by L being indifferent to substitutions we have that $\mu(\varphi_0) \wedge \cdots \wedge \mu(\varphi_{n-1}) \vdash_L \mu(\bot)$ and clearly $\mu(\bot) \vdash_L \bot$ so we have by z being deductively closed that $\bot \in z$, a contradiction.

□

Even though $h_{\mathcal{U}}$ is a map between canonical frames over two different languages, it is conservative, in the sense that if we restrict ourselves to formulae in $\mathcal{S}(P)$ when we look at the makeup of the members of X_Q, $h_{\mathcal{U}}$ will not effect any change:

Proposition 8.5. *For $\mathcal{U}$ as above, $h_{\mathcal{U}}(z) \cap \mathcal{S}(P) = z$.*

Proof. We show that $z \subseteq h_{\mathcal{U}}(z)$ and, by the maximal consistency of both these sets, we will get our result. So let $\varphi \in z$. Thus for all $\mu \in \Omega$, $\mu(\varphi) = \varphi$ by $\mu \restriction P = \mathrm{id}$ and trivially $\{\mu \in \Omega \mid \mu(\varphi) \in z\} = \Omega \in \mathcal{U}$ giving us that $\varphi \in h_{\mathcal{U}}(z)$. □

Our construction will be complete when we verify that $h_{\mathcal{U}}$ is indeed accessibility preserving.

Theorem 8.6. *For $\mathcal{U}$ as before, $h_{\mathcal{U}}$ is an accessibility preserving injection from F_P into F_Q.*

Proof. That $h_{\mathcal{U}}$ is one-one is an immediate consequence of the previous proposition, so all that remains to be shown is that

$$(\forall i \in Idx)(\forall z_1, z_2 \in X_P)\left[\langle z_1, z_2\rangle \in R_{Pi} \Longleftrightarrow \langle h_{\mathcal{U}}(z_1), h_{\mathcal{U}}(z_2)\rangle \in R_{Q_i}\right].$$

So let $i \in Idx$, $z_1, z_2 \in X_P$.

The *if* part is straightforward:

$$\begin{aligned}\langle h_{\mathcal{U}}(z_1), h_{\mathcal{U}}(z_2)\rangle \in R_{Q_i} &\Longleftrightarrow (\forall \varphi \in \mathcal{S}(Q))\left[\Box_i\varphi \in h_{\mathcal{U}}(z_1) \Longrightarrow \varphi \in h_{\mathcal{U}}(z_2)\right]\\ &\Longrightarrow (\forall \varphi \in \mathcal{S}(P))\left[\Box_i\varphi \in h_{\mathcal{U}}(z_1) \Longrightarrow \varphi \in h_{\mathcal{U}}(z_2)\right]\\ &\Longleftrightarrow (\forall \varphi \in \mathcal{S}(P))\left[\Box_i\varphi \in z_1 \Longrightarrow \varphi \in z_2\right]\\ &\Longleftrightarrow \langle z_1, z_2\rangle \in R_{Pi}.\end{aligned}$$

with the second line following by restriction, and the third by the previous proposition.

For the *only if* part, suppose that $i \in Idx$, $\langle z_1, z_2\rangle \in R_{Pi}$, and let $\Box_i\varphi \in h_{\mathcal{U}}(z_1)$. By $\langle z_1, z_2\rangle \in R_{Pi}$ we have that

$$\{\mu \in \Omega \mid \mu(\Box_i\varphi) \in z_1\} = \{\mu \in \Omega \mid \Box_i\mu(\varphi) \in z_1\} \subseteq \{\mu \in \Omega \mid \mu(\varphi) \in z_2\}.$$

Since the first of these sets is in $\mathcal{U}$, the last one must also be in $\mathcal{U}$ giving $\varphi \in h_{\mathcal{U}}(z_2)$. □

When $\mathcal{U}$ is a principal ultrafilter, $h_{\mathcal{U}}$ will correspond to a standard map, and this may also be the case for particular non-principal ultrafilters. To see that $h_{\mathcal{U}}$ can in fact be a non-standard map is relatively straightforward, however we will leave that discussion till near the end of this chapter at which point the theoretical development will make such a result almost obvious.

8.4 Finding Variables

To meet our promise of providing an embedding which hits particular points we will adopt a construction closely modelled on that used in the proof of the downward Löwenheim-Skolem theorem.

Our approach will be to take each n-tuple of points that we wish to include (remember that each point is a maximal consistent set of formulae) and extract from those points the formulae, and thus the propositional variables, which contribute to the 'variability' in, and the 'structure' of, the n-tuple.

Lemma 8.7. *Given $P \subseteq Q$ and $\overline{x} \in X_Q$, there is a set $e(P, \overline{x}) \subseteq Q$ such that:*

1. *For each $\overline{\varphi} \in \mathcal{S}(Q)$ (with $\overline{\varphi}$ and $\overline{x}$ having the same arity n), there is a $\mu \in \Omega_{e(P,\overline{x}),Q}$ so that*
$$(\forall j \leqslant n)\left[\varphi_j \in x_j \iff \mu(\varphi_j) \in x_j\right].$$

2. $\operatorname{card}(e(P, \overline{x})) \leqslant \max(\omega, \operatorname{card}(\mathit{Idx}), \operatorname{card}(P))$.

Proof. We need only look at the case where $\omega < \operatorname{card}(Q)$ and $\operatorname{card}(P) < \operatorname{card}(Q)$. We will construct the set $e(P, \overline{x})$ in stages. Let S be a countable set of new propositional variables distinct from those in Q. Let $P_0 = P$ and suppose that P_k has already been constructed. Let

$$\Theta_k = \{\mu : P_k \cup S \longrightarrow Q \mid \mu \text{ is constant on } P_k, \mu \text{ is one-one}\},$$

let $\overline{\varphi} \in \mathcal{S}(P_k \cup S)$, and let $\sim^k_{\overline{\varphi}}$ be an equivalence relation on Θ_k defined by

$$\mu \sim^k_{\overline{\varphi}} \nu \iff (\forall j \leqslant n)\left[\mu(\varphi_j) \in x_j \iff \nu(\varphi_j) \in x_j\right].$$

There can be at most 2^n equivalence classes since $\mu \sim^k_{\overline{\varphi}} \nu$ iff they have the same 'signature,' where $j \in \operatorname{sig}\mu$, the signature of μ, iff $\mu(\varphi_j) \in x_j$, and clearly $\operatorname{sig}\mu \subseteq \mathcal{P}(n)$.

So let $\Delta^{\overline{\varphi}}_k \subseteq \Theta_k$ be a set of size $\operatorname{card}\left(\Theta_k / \sim^k_{\overline{\varphi}}\right) \leqslant 2^n$, with each element a representative of a distinct $\sim^k_{\overline{\varphi}}$ equivalence class. Set

$$P_{k+1} = P_k \cup \bigcup \left\{\operatorname{range}(\mu) \mid \mu \in \Delta^{\overline{\varphi}}_k, \overline{\varphi} \in \mathcal{S}(P_k \cup S)\right\}.$$

For each $\mu \in \Delta^{\overline{\varphi}}_k$ the size of $\operatorname{range}(\mu)$ is bounded by $\operatorname{card}(P_k) + \operatorname{card}(S)$, and there are $(\omega + \operatorname{card}(\mathit{Idx}) + \operatorname{card}(P_k) + \operatorname{card}(S))^n$ sequences $\overline{\varphi}$ of length n in $\mathcal{S}(P_k \cup S)$, thus

$$\begin{aligned}\operatorname{card}(P_{k+1}) &\leqslant \operatorname{card}(P_k) + 2^n(\omega + \operatorname{card}(\mathit{Idx}) + \operatorname{card}(P_k) + \operatorname{card}(S))^{n+1} \\ &\leqslant \max(\operatorname{card}(P_k), \operatorname{card}(\mathit{Idx}), \omega).\end{aligned}$$

So, put $P_\omega = \bigcup_{k \in \omega} P_k$ and we complete the proof of this lemma by making the claim that we can put $e(P, \overline{x}) = P_\omega$. The proof of this claim consists of two verifications:
$(\forall \overline{\varphi} \in \mathcal{S}(Q))\,(\exists \mu \in \Omega_{P_\omega, Q})\,(\forall j \leqslant n)\left[\varphi_j \in x_j \iff \mu(\varphi_j) \in x_j\right]$.

Let $\overline{\varphi} \in \mathcal{S}(Q)$. Since $\overline{\varphi}$ consists of a finite sequence of finitely long formulae we can assume that $\overline{\varphi} \in \mathcal{S}(P_k \cup (Q - P_\omega))$ for some $k \in \omega$. Clearly, there is a $\overline{\psi} \in \mathcal{S}(P_k \cup S)$ and a one-one function $\nu_1 : P_k \cup S \longrightarrow Q$, identity on P_k, so that $\nu_1(\overline{\psi}) = \overline{\varphi}$. Further, ν_1 has a representative $\nu_2 \in \Delta^{\overline{\varphi}}_k$ in its $\sim^k_{\overline{\varphi}}$-equivalence class.

Now, let $\mu \in \Omega_{P_\omega, Q}$ be defined by

$$\mu(q) = \begin{cases} q & \text{if } q \in P_\omega \\ \nu_2 \circ \nu_1^{-1}(q) & \text{if } q \in \text{range}(\nu_1) \\ p_0 & \text{otherwise} \end{cases}$$

where p_0 is any element of P_ω. We have that

$$\begin{aligned} \mu(\varphi_j) \in x_j &\Longleftrightarrow \nu_2 \circ \nu_1^{-1}(\varphi_j) \in x_j \\ &\Longleftrightarrow \nu_2(\psi_j) \in x_j \\ &\Longleftrightarrow \nu_1(\psi_j) \in x_j \Longleftrightarrow \varphi_j \in x_j \end{aligned}$$

with the second to last line following by $\nu_1 \sim^k_{\overline{\varphi}} \nu_2$.

$\underline{\text{card}(P_\omega) \leqslant \max(\text{card}(P), \text{card}(\mathit{Idx}), \omega)}$

This follows by our observation that

$$\text{card}(P_{k+1}) \leqslant \max(\text{card}(P_k), \text{card}(\mathit{Idx}), \omega).$$

□

The above lemma shows that given an n-tuple of points it is possible to find a collection of propositional variables so that the formulae over those propositional variables adequately 'characterize' all the formulae resident in those points, and that this set of propositional variables is not too large.

We will be ready for the next section when we show that we can find a similar collection of propositional variables which 'characterize' any set of points in a canonical frame.

Lemma 8.8. *Given $P \subseteq Q$ and $Y \subseteq X_Q$, there is a set $e(P, Y) \subseteq Q$ such that:*

1. *For each $\overline{\varphi} \in \mathcal{S}(Q)$ and $\overline{x} \in Y$ (each having the same arity n), there is a $\mu \in \Omega_{e(P,Y),Q}$ so that*

$$(\forall j \leqslant n)\,[\varphi_j \in x_j \Longrightarrow \mu(\varphi_j) \in x_j].$$

2. $\text{card}(e(P, Y)) \leqslant \max(\omega, \text{card}(\mathit{Idx}), \text{card}(P), \text{card}(Y))$.

Proof. Iterate the operation $P \mapsto \bigcup_{\overline{x} \in Y} e(P, \overline{x})\ \omega$ many times. □

8.5 Specific Target Embeddings

We are now able to piece together the two previous sections to produce an accessibility preserving embedding from $X_{P'}$ to X_Q, for some $P' \subseteq Q$ of restricted cardinality, which is guaranteed to hit some pre-specified subset of X_Q.

To that end, fix $P \subseteq Q$, $Y \subseteq X_Q$, set $P' = e(P, Y)$ and let $\Omega = \Omega_{P',Q}$. For each $\overline{\varphi} \in \mathcal{S}(Q)$ and $\overline{x} \in Y$ of the same arity n, define

$$\Gamma_{\overline{\varphi},\overline{x}} = \{\mu \in \Omega \mid (\forall j \leqslant n)\,[\mu(\varphi_j) \in x_j \iff \varphi_j \in x_j]\}.$$

Lemma 8.9. $\{\Gamma_{\overline{\varphi},\overline{x}} \mid \overline{\varphi} \in \mathcal{S}(Q), \overline{x} \in Y, \overline{\varphi}$ *and* $\overline{x}$ *have the same arity* $\}$ *can be extended to an ultrafilter* $\mathcal{U}$.

Proof. That each $\Gamma_{\overline{\varphi},\overline{x}}$ is non-empty can be seen directly from Lemma 8.8, so it is sufficient to show that for $\overline{\varphi}, \overline{x}$ and $\overline{\psi}, \overline{y}$ there is a $\overline{\chi}, \overline{z}$ so that $\Gamma_{\overline{\varphi},\overline{x}} \cap \Gamma_{\overline{\psi},\overline{y}} \supseteq \Gamma_{\overline{\chi},\overline{z}}$. Let $\overline{\chi} = \overline{\varphi}^\frown\overline{\psi}$ and $\overline{z} = \overline{x}^\frown\overline{y}$ and suppose that $\mu \in \Gamma_{\overline{\chi},\overline{z}}$. We have two verifications:

Case $\mu \in \Gamma_{\overline{\varphi},\overline{x}}$

Thus, for all $j < \text{length}(\overline{\varphi})$,

$$\begin{aligned} \mu(\varphi_j) \in x_j &\iff \mu(c_j) \in z_j \\ &\iff \varphi_j \in x_j. \end{aligned}$$

Case $\mu \in \Gamma_{\overline{\psi},\overline{y}}$

Similar.

□

Theorem 8.10. *For* $Y \subseteq X_Q$ *and* $P \subseteq Q$ *there exists a* P', $P \subseteq P' \subseteq Q$ *with*

$$\text{card}(P') \leqslant \max(\text{card}(Y), \text{card}(P), \text{card}(Cnct), \omega)$$

so that there is an ultrafilter embedding

$$h_{\mathcal{U}} : F_{P'} \longrightarrow F_Q$$

with $h_{\mathcal{U}}(y \cap \mathcal{S}(P')) = y$ *for all* $y \in Y$ *and so* $\text{range}(h_{\mathcal{U}}) \supseteq Y$.

Proof. Let Ω be as given at the start of this section and $\mathcal{U}$ as given by the previous lemma. All that remains to be shown is that $h_{\mathcal{U}}(y \cap \mathcal{S}(P')) =$

y for each $y \in Y$. So let $\varphi \in \mathcal{S}(Q)$. We know by definition that $(\forall \mu \in \Gamma_{\varphi,y})\,[\mu(\varphi) \in y \Longleftrightarrow \varphi \in y]$ and so

$$\begin{aligned}\varphi \in y &\Longrightarrow (\forall \mu \in \Gamma_{\varphi,y})\,[\mu(\varphi) \in y] \\ &\Longrightarrow \mu(\varphi) \in y \quad \mathcal{U}\text{ a.e.} \\ &\Longrightarrow \mu(\varphi) \in y \cap \mathcal{S}(P') \quad \mathcal{U}\text{ a.e.}\end{aligned}$$

From which we conclude that $y \subseteq h_{\mathcal{U}}(y \cap \mathcal{S}(P'))$ and maximal consistency then gives us our result. □

8.6 There Are Non-standard Embeddings

The existence of a non-standard embedding can now be seen through the use of an observation of HAJNAL ANDRÉKA (reported to the author in [30]), however we must first isolate a class of common logics.

Definition 8.11. A logic is said to *admit arbitrarily many distinguishable alternatives* in index $i \in Idx$ iff for each $n \in \omega$ there exists a model $M = (X, \underline{R}, V)$ for that logic, points $x, y_j \in X$, $j < n$, and formulae φ_j, $j < n$ such that for all $j, k < n$, $\langle x, y_k\rangle \in R_i$, and $M \vDash_{y_k} \varphi_j$ if and only if $k = j$.

Theorem 8.12 (ANDRÉKA). *Suppose that L is a logic which admits arbitrarily many distinguishable alternatives in index $i \in Idx$ and that* $\mathrm{card}(Q) > \mathrm{card}(P) \geqslant \omega$. *Then there is a point in X_Q which is not in the image of any standard homomorphism from F_P to F_Q.*

Proof. Let $y \in X_Q$ be a maximal consistent extension of

$$\{\Diamond_i(p \wedge \neg q) \mid p, q \in Q, p \neq q\}.$$

Because L admits arbitrarily many distinguishable alternatives this set is indeed consistent.

Assume that $y = g_+(x)$ for some $x \in X_P$ and some $g : \mathcal{S}(Q) \longrightarrow \mathcal{S}(P)$. Since $\mathrm{card}(Q) > \mathrm{card}(P) \geqslant \omega$, $g(q) = g(p)$ for some $p, q \in Q$. Thus $\vdash_L \neg(g(p) \wedge \neg g(q))$ and so by necessitation $\vdash_L \neg\Box_i(g(p) \wedge \neg g(q))$. This tells us that $\vdash_L \neg g(\Box_i(p \wedge \neg q))$, giving $g(\Box_i(p \wedge \neg q)) \notin x$, and we can then conclude that $\Box_i(p \wedge \neg q) \notin g_+(x) = y$, a contradiction. □

Since our construction shows that for most logics there is an ultrafilter embedding which will hit the y constructed in our proof above, we know that such an embedding will indeed be non-standard.

The reader should be reminded that these ultrafilter embeddings are by no means homomorphisms and to the author's knowledge the existence of non-standard homomorphisms is still a big question—answered only in very specific cases in this thesis. However, these non-standard ultrafilter embeddings will sometimes be the best we can do.

Consider this extension of ANDRÉKA's argument: Suppose that $2^{\mathrm{card}(Q)} > 2^{\mathrm{card}(P)}$ holds[2] and that L is a logic which admits arbitrarily many distinguishable alternatives in index $i \in Idx$. Set y to be a maximal consistent extension of

$$\left\{\Diamond\left(\neg^{k_0}p_0 \wedge \cdots \wedge \neg^{k_{n-1}}p_{n-1}\right) \mid (k_0, \ldots, k_{n-1}) \in {}^2n,\ n \in \omega,\ p_0, \ldots, p_{n-1} \in Q\right\}$$

where we take $\neg^0 = \neg\neg$ and $\neg^1 = \neg$.

As with ANDRÉKA's argument the above set is clearly finitely consistent and y must have $2^{\mathrm{card}(Q)}$ many alternatives because for each subset W of Q there must be a y alternative which affirms only the variables of W and no others. In this case, there is no way that y could be in a standard or non-standard frame homomorphic image of F_P (since $2^{\mathrm{card}(Q)}$ y-alternatives will be in that same image) and so if we are really intent on hitting y, the ultrafilter embedding may well be the limit of what we can hope.

8.7 Logics of Bounded Alternative

In this section we will see that we can actually get homomorphisms if we are dealing with logics of bounded alternative. This will allow us to conclude, in the next chapter, that there are a host of non-standard frame homomorphims and even non-standard frame isomorphisms and automorphisms for these logics.

While we have so far taken great pains to point out that ultrafilter maps are not frame homomorphisms, and that this can sometimes be unavoidable, there is a notable exception where the logic L is of bounded alternative:

Theorem 8.13. *Suppose that* $\mathrm{Alt}_n^i(\overline{p}) \in L$ *for some* $n \in \omega$ *and* $i \in Idx$. *Then* $h_{\mathcal{U}} : F_P^L \longrightarrow F_Q^L$ *is a frame homomorphism in index* i.

Proof. We already have that $xR^L_{P_i}y \Longrightarrow h_{\mathcal{U}}(x)\, R^L_{Q_i} h_{\mathcal{U}}(y)$, so we are left with the second condition of Definition 3.35.

Suppose that $h_{\mathcal{U}}(x)\, R^L_{Q_i} y$. Let $\{x_j \mid j < n\} = R^L_{P_i}(x)$ (since L is a logic of bounded alternative in index i we can do this without any trouble) and take

[2] Again, we have the question of whether anything special happens when $\mathrm{card}(Q) > \mathrm{card}(P)$ but $2^{\mathrm{card}(Q)} = 2^{\mathrm{card}(P)}$. See Chapter 9.

$y \neq h_{\mathcal{U}}(x_j)$ for all $j < n$ or else we would be done. Since $\{h_{\mathcal{U}}(x_j) \mid j < n\}$ is finite we can find a $\varphi \in \mathcal{S}(Q)$ so that $\varphi \in y$ and $(\forall j < n)\,[\varphi \notin h_{\mathcal{U}}(x_j)]$. Thus $\Diamond_i\varphi \in h_{\mathcal{U}}(x)$ and $\neg\varphi \in h_{\mathcal{U}}(x_j)$ for all $j < n$, giving

$$U_j := \{\mu \mid \mu(\neg\varphi) \in x_j\} \in \mathcal{U} \quad \text{for } j < n, \text{ and}$$
$$U_n := \{\mu \mid \mu(\Diamond\varphi_i) \in x\} \in \mathcal{U}.$$

By $\mathcal{U}$ having the finite intersection property $U := U_0 \cap \cdots \cap U_n \in \mathcal{U}$ and we can conclude that there is a $\mu_0 \in \Omega_{P,Q}$ so that $\mu_0 \in U$. We know that $\mu_0(\neg\varphi) \in x_j$ for all $j < n$ and $\mu_0(\Diamond_i\varphi) \in x$, so $\neg\mu_0(\varphi) \in x_j$ for all $j < n$ and $\Diamond_i\mu_0(\varphi) \in x$. But $xR^L_{P_i}x' \iff (\exists j < n)\,[x' = x_j]$ and so $\Box\neg\mu_0(\varphi) \in x$, contradicting $\Diamond_i\mu_0(\varphi) \in x$. $\square$

It is worth noting that our proof really relies on the fact that we can separate the closed set $\{h_{\mathcal{U}}(x_j) \mid j < n\}$ from the point y, and so if $h_{\mathcal{U}}\left[R^L_{P_i}(x)\right]$ were guaranteed to be closed then we could get the same result. We then get the following corollary to the proof:

Corollary 8.14. *The function $h_{\mathcal{U}}$ is a frame homomorphism if it is a closed map.*

We include the next result since it is an easy consequence of our development so far. This result was suggested by GOLDBLATT in [30] and implies that all logics of bounded alternative have the property that canonicity in any infinite cardinal proves canonicity in all infinite cardinals. The conclusion is not so astounding as it is known that all logics of bounded alternative are canonical (see FABIO BELLISSIMA's [5]), but the result below does highlight a success of the approach of relating canonical frames of different cardinality.

Corollary 8.15. *For L a logic of bounded alternative in each index $i \in Idx$, and for each $\lambda < \kappa$ there is a disjoint union of copies of F^L_λ which has F^L_κ as a frame homomorphic image.*

Proof. For each $y \in X^L_\kappa$, let (F^L_λ, y) be a disjoint copy of F^L_λ and $h_{\mathcal{U}_y} : (F^L_\lambda, y) \longrightarrow F^L_\kappa$ be an ultrafilter embedding which hits y. By Theorem 8.10 such a map can be found. By Theorem 8.13 each such map is a frame homomorphism (in each index $i \in Idx$). Let F be the disjoint union of the (F^L_λ, y), $y \in X^L_\kappa$ and h the disjoint union of the $h_{\mathcal{U}_y}$. Then h is a frame homomorphism onto F^L_κ. $\square$

Chapter 9

Isomorphisms Between Canonical Frames

9.1 Introduction

In the last chapter we looked at a basic type of map between canonical frames and saw that there are many non-standard relationships between canonical frames of different cardinalities. In this chapter we will go further and look at the question of which conditions ensure that F^L_λ and F^L_κ are isomorphic. We know from Proposition 2.37 that these canonical frames have cardinalities 2^λ and 2^κ respectively and so the only way that they can be isomorphic is to posit that $2^\lambda = 2^\kappa$. We will look at what happens when we make this assumption to see if we can actually have the isomorphism hold and we will look at some relatively simple systems to see what is happening there.

Section 9.2 will show that canonical frames really are complicated, and are full of complicated subcomponents. Section 9.3 will show us that this has important consequences for the logic S5 and logics below it, Section 9.4 will provide a start to the characterisation of logics which admit these non-standard isomorphisms, and the last section, 9.5, will show that for the friendly logics of bounded alternative we can have little difficulty in finding non-standard automorphisms and isomorphisms.

9.2 Disjoint Generated Subframes and Closed Sets

Our first goal will be a look at the possibility that $F^L_\omega \cong F^L_{\omega_1}$, and to see that this cannot possibly hold for logics 'below' S5; this will follow clearly from the possible sizes of certain generated subframes of the canonical frame. Logics which are contained in S5 (and thus have canonical frames which contain the canonical frame of S5) are natural for this because of the simplicity of their equivalence classes and inspection of the upcoming proofs will show that we rely heavily on this simplicity to the point where the S5 properties seem almost necessary.

Before looking at S5 though, we need a few easy but general results about generated subframes and closed sets.

Lemma 9.1. *F_κ^L contains 2^κ pairwise disjoint generated subframes isomorphic to F_κ^L.*

Proof. Write $F = F_\kappa^L$, $X = X_\kappa^L$. Let $\kappa = E_1 \cup E_2$ where $E_1 \cap E_2 = \varnothing$, $\operatorname{card}(E_1) = \operatorname{card}(E_2) = \kappa$ witnessed by the bijection $g : E_2 \longrightarrow \kappa$. For each $\Delta \subseteq E_1$ define $g^\Delta : \kappa \longrightarrow \mathcal{S}(\kappa)$ by

$$g^\Delta(\alpha) = \begin{cases} g(\alpha) & \text{if } \alpha \in E_2 \\ \top & \text{if } \alpha \in \Delta \\ \bot & \text{if } \alpha \notin \Delta \end{cases}$$

Now, g^Δ is onto κ since $g^\Delta \restriction E_2 = g$. Thus $g_+^\Delta : F \longrightarrow F$ is into. Also, g_+^Δ is a frame homomorphism so $g_+^\Delta[X]$ is a generated subframe of F isomorphic to F.

For $\Delta_1 \neq \Delta_2$, $g_+^{\Delta_1}[X] \cap g_+^{\Delta_2}[X] = \varnothing$.

Without loss of generality let $\alpha \in \Delta_1 - \Delta_2$, so $\alpha \in E_1$. Thus $g^{\Delta_1}(\alpha) = \top$, $g^{\Delta_2}(\alpha) = \bot$. Thus

$$(p_\alpha \leftrightarrow \top) \in \bigcap g_+^{\Delta_1}[X] \qquad (p_\alpha \leftrightarrow \bot) \in \bigcap g_+^{\Delta_2}[X]$$

Assume that $g_+^{\Delta_1}[X] \cap g_+^{\Delta_2}[X] \neq \varnothing$. Thus there is an x with $(p_\alpha \leftrightarrow \top), (p_\alpha \leftrightarrow \bot) \in x$. So, $\top \leftrightarrow \bot \in x$ a contradiction.

Thus we have established that these are all disjoint generated subframes. □

Corollary 9.2. *If G is a generated subframe of F_κ^L then there are 2^κ many pairwise disjoint generated subframes of F_κ^L each isomorphic to G.*

Lemma 9.3. *Suppose that λ is a cardinal with $\omega < \lambda < 2^\omega$. Then there is no closed subset of X_ω^L of size λ.*

Proof. Firstly, we need only consider the case where λ is a regular cardinal. In the case that λ is a limit cardinal we just note that this implies that there is a regular cardinal between ω and λ with which we can work.

Assume Y is a closed subset of X_ω^L of size greater than or equal to λ. We will show the following:

Let φ be a formula so that $\operatorname{card}(Y \cap \|\varphi\|) \geqslant \lambda$, then there is a formula ψ so that both

$$\operatorname{card}(Y \cap \|\varphi \wedge \psi\|) \geqslant \lambda \leqslant \operatorname{card}(Y \cap \|\varphi \wedge \neg\psi\|).$$

This will give our result since it will mean that Y can be sectioned into successively finer parts and each of the 2^ω many descending chains of successively finer sections has the finite intersection property and so has a non-empty intersection in Y (distinct from other sections) and so Y must have size 2^ω.

So, suppose $Y \cap \|\varphi\|$ is of a size greater than or equal to λ. Assume that for each ψ, one of

$$\operatorname{card}(Y \cap \|\varphi \wedge \psi\|) < \lambda \text{ or } \operatorname{card}(Y \cap \|\varphi \wedge \neg\psi\|) < \lambda.$$

Let $\langle \psi_j \mid j \in \omega \rangle$ be so that $\{\psi_j, \neg\psi_j \mid j \in \omega\}$ exhausts[1] all of $\mathcal{S}(\omega)$, and for each j,

$$\operatorname{card}(Y \cap \|\varphi \wedge \psi_j\|) < \lambda \text{ so } \operatorname{card}(Y \cap \|\varphi \wedge \neg\psi_j\|) \geqslant \lambda.$$

Thus we have that

$$\begin{aligned}
&\bigcup_j (Y \cap \|\varphi \wedge \psi_j\|) \cup \bigcap_j (Y \cap \|\varphi \wedge \neg\psi_j\|) \\
&= Y \cap \|\varphi\| \cap \left(\bigcup_j \|\psi_j\| \cup \bigcap_j \|\neg\psi_j\| \right) \\
&= Y \cap \|\varphi\| \cap \bigcap_j \left(\|\neg\psi_j\| \cup \bigcup_k \|\psi_k\| \right) \\
&= Y \cap \|\varphi\| \cap \bigcap_j X^L_\omega = Y \cap \|\varphi\|
\end{aligned}$$

Since

1. $(\forall j \leqslant \omega)\,[\operatorname{card}(Y \cap \|\varphi \wedge \psi_j\|) < \lambda]$,
2. $\operatorname{card}(Y \cap \|\varphi\|) \geqslant \lambda$, and
3. λ is a regular cardinal

we have that $\operatorname{card}\left(\bigcap_j (Y \cap \|\varphi \wedge \neg\psi_j\|)\right) \geqslant \lambda$. But $\{\psi_j, \neg\psi_j | j \in \omega\}$ exhausts all of $\mathcal{S}(\omega)$ and so this intersection must have cardinality at most 1. □

This result is really just a result of set theoretic topology (see, e.g., I. JUHASZ's article [46, Theorem 4.4 p. 84], which gives a slightly more general result). In [55] K. KUNEN[2] asks if this topological result can be generalized to the cardinal ω_1 and he reported that such a conclusion, while implied

[1] Modulo the equivalence of φ and $\neg\neg\varphi$.

[2] See [46] for a presentation of this work.

by certain large cardinal hypotheses, can be falsified in some models of ZFC.[3] This work tells us that our result above, and the conclusions about systems contained in S5 which follow, cannot be generalised to higher cardinals.

9.3 Implications for the logic S5

Here we will see that the previous two Lemmas identify all the possible sizes of generated subframes of canonical frames associated with S5 and this will allow us to conclude that $F^L_\omega \cong F^L_{\omega_1}$ does not hold in general.

Since our formalism is based around multi-modal logics we will deal with a logic that we will know as *super* S5, or sS5 for short. It has axioms (where the i and j range over all of *Idx*):

1. $\Box_i \top$,
2. $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$,
3. $\Box_i p \to p$,
4. $\Box_i p \to \Box_i \Box_i p$, and
5. $\Box_i p \leftrightarrow \Box_j p$.

The first four axioms say that each modality is S5-like and so its frames have equivalence relations as their accessibility relations, and the last axiom is extremely severe, requiring that all accessibility relations are identical. Thus a typical frame for sS5 is $(Y, \underline{Y \times Y})$, where $\underline{Y \times Y}$ is the sequence of relations whose components are uniformly equal to $Y \times Y$.

This logic may seem highly specialised and restrictive, however for our purposes this does not matter as Lemma 9.5, and so the results of this section, are about logics which are *contained in* sS5, so specialisation of this upper bound cannot make our result any less general.

Lemma 9.4. *If $(Y, \underline{Y \times Y})$ is a generated subframe of F^L_κ then Y is a closed subset of X^L_κ.*

Proof. Let $y_0 \in Y$ be any point and let $i \in Idx$. Then $Y = \|\{\varphi \mid \Box_i \varphi \in y_0\}\|$. $\Box$

Lemma 9.5. *Suppose that $L \subseteq$ sS5 and that λ is a cardinal with $\lambda \leqslant \kappa$. Then there is a generated subframe of F^L_κ of size λ, and thus 2^κ many generated subframes of size λ.*

[3] This topological analysis, it turns out, can be reduced to an analysis of trees, and in particular it reduces to a question about the existence of Jech-Kunen trees.

Proof. Consider the following set of formulae:

$$\Sigma = \mathsf{sS5} \cup \{\Box_i(\neg p_\alpha \to p_\beta),\ \Diamond_i(p_\alpha \wedge \neg p_\beta) \mid \alpha \neq \beta < \lambda, i \in \mathit{Idx}\}$$
$$\cup \{\Box_i p_\alpha \mid \lambda \leqslant \alpha < \kappa,\ i \in \mathit{Idx}\}.$$

The set Σ is (sS5-consistent and hence) L-consistent since $(\lambda, \underline{\lambda \times \lambda}, v)$ is an sS5 model of Σ where $v(p_\alpha) = \lambda - \{\alpha\}$.

Let $y_0 \in \|\Sigma\|$, within the canonical frame F_κ, and let Y be the subframe of F_κ generated by y_0. Let $\underline{R} = \underline{Y \times Y}$. Thus $(Y, \underline{R})$ is a generated subframe of F_κ. Now, $Y \cap \|\neg p_\alpha\|$ has cardinality 1 for if $y_1, y_2 \in Y \cap \|\neg p_\alpha\|$ for distinct y_1, y_2 we have that there is a propositional variable p_β which distinguishes them (remember that this is an sS5 equivalence) so $\beta < \lambda$, but $\neg p_\alpha \in y_1 \cap y_2$ so $p_\beta \in y_1 \cap y_2$. Clearly

$$Y = \left(\bigcup_\alpha (Y \cap \|\neg p_\alpha\|)\right) \cup \left(\bigcap_\alpha (Y \cap \|p_\alpha\|)\right).$$

Since Y is an sS5-equivalence class each element is determined solely by its propositional variables so $\operatorname{card}(Y \cap \bigcap_\alpha \|p_\alpha\|) = 1$. Thus, $\operatorname{card}(Y) \leqslant \lambda + 1 = \lambda$. But, $\lambda \leqslant \operatorname{card}(Y)$ since $\{\Diamond_i(p_\alpha \wedge \neg p_\beta) \mid \alpha \neq \beta < \lambda\} \subseteq \Sigma$, giving $\operatorname{card}(Y) = \lambda$. □

Remark 9.6. This result might hold for more general logics but the author is unable to see how this could be shown. Certainly the approach adopted here will not work as it relied heavily on two points being distinguishable by a propositional variable and this can only be guaranteed when those points are in a 'super' equivalence class of the frame.

Theorem 9.7. *For any logic* $L \subseteq \mathsf{sS5}$, $F^L_\omega \not\cong F^L_{\omega_1}$.

Proof. If $2^{\omega_1} > 2^\omega$ we are done so assume $2^{\omega_1} = 2^\omega$. By Lemmas 9.1 and 9.5 $F^L_{\omega_1}$ contains a generated subframe of size ω_1 and so $\omega < \omega_1 < 2^\omega$. By Lemma 9.3, F^L_ω does not contain such a generated subframe (or else, by Lemma 9.4, X^L_ω would have a closed set of that size) and so they cannot be isomorphic. □

Applying Scroggs's theorem (a result of SCROGGS's paper [71]) we see quite quickly that, S5 is an upper limit and the desired result can go through above it.

Theorem 9.8. *Suppose that L is a mono-modal logic, i.e.,* $\operatorname{card}(\mathit{Idx}) = 1$. *If* $L \supsetneq \mathsf{S5}$ *then* $F^L_\omega \cong F^L_{\omega_1}$ *in the presence of* $2^\omega = 2^{\omega_1}$.

Proof. By Scroggs's theorem L must be a logic whose frames have equivalence classes of size n or less, for some finite n. Thus, for each $m \leqslant n$ the canonical frames F^L_ω and $F^L_{\omega_1}$ must each have 2^ω and 2^{ω_1} equivalence classes of size m respectively. Since we are dealing with such a simple structure and $2^\omega = 2^{\omega_1}$ we have that $F^L_{\omega_1} \cong F^L_\omega$. □

Remark 9.9. The above theorem suggest that a similar result holds for logics of bounded alternative and in Section 9.5 we use the theory of ultrafilter embeddings to deduce just that.

We will leave S5 for the moment returning to it when we investigate automorphisms.

9.4 Non-Standard Isomorphisms and Automorphisms

When the author originally approached the question of whether $F^L_\omega \cong F^L_{\omega_1}$ it was clear that if it holds, then it must be witnessed by a non-standard isomorphism. In this section we show that such non-standard maps are often not present in logics satisfying the condition given below, not only allowing us to conclude that $F^L_\omega \ncong F^L_{\omega_1}$ yet again, but also the stronger result that such logics have no non-standard automorphisms.

Definition 9.10. We say that L *exhibits all consistencies in index* i, for some $i \in$ *Idx* iff, for each propositional variable, $p \in \omega$

$$\{\Box_i p\} \cup \{\Diamond_i \varphi \mid p \wedge \varphi \in \mathcal{S}(P) \text{ is } L\text{-consistent}\}$$

is consistent.

Note that by a logic being closed under substitution we can conclude that, as long as it is infinite, the exact nature of the underlying set of propositional letters, in this case ω, is irrelevant to this definition.

Example 9.11. The logic S5 does not exhibit all consistencies as $p \wedge \Diamond\neg p$ is S5-consistent, however $\Box p \wedge \Diamond(p \wedge \Diamond\neg p)$ is not consistent with S5.

Remark 9.12. A (stronger) condition which will guarantee that L exhibits all consistencies is that in any model $M = (X, \underline{R}, v)$ for L and for any $Y \subseteq X$ there is a new model $M' = (X', \underline{R'}, v')$ for L such that M is a generated submodel of M' and $(\exists x' \in X')\,[R'_i(x') = Y]$. Some non-iterative logics such as K, and T are immediate examples of logics satisfying this condition.

Example 9.13. KMcK satisfies all consistencies but is not contained in S5.

Proof. We use a modification of the condition mentioned above. Let $M = (X, R, v)$ be the canonical model for KMcK. We suppress the super and subscripts assuming that it is the canonical model over ω. Let $p \in \omega$ and let $x' \notin X$. Define

$$\begin{aligned} X' &= X \cup \{x\}, \\ R' &= R \cup \{\langle x', y\rangle \mid y \in X, p \in y\}, \text{ and} \\ v' &= v. \end{aligned}$$

Set $M' = (X', R', v')$. If we can show that M' is a model for KMcK we are done with the first part as $p \wedge \varphi$ consistent implies $\langle x', y\rangle \in R$ for any y with $p \wedge \varphi \in y$, so $M' \vDash_{x'} \Diamond\varphi$. Also, it is clear that $M' \vDash_{x'} \Box p$. Since M is a generated submodel of M', the only place where a problem might occur is x' so we just need to test whether McK holds at x'. Suppose $M' \vDash_{x'} \Box\Diamond\varphi$. We have,

$$\begin{aligned} M' \vDash_{x'} \Box\Diamond\varphi &\Longrightarrow (\forall y \in X')\,[\langle x', y\rangle \in R' \Longrightarrow \Diamond\varphi \in y] \\ &\Longrightarrow (\forall y \in X)\,[p \in y \Longrightarrow \Diamond\varphi \in y] \\ &\Longrightarrow (\forall y \in X)\,[p \to \Diamond\varphi \in y] \\ &\Longrightarrow p \to \Diamond\varphi \in \mathsf{KMcK}. \end{aligned}$$

Now, assume $M' \nvDash_{x'} \Diamond\Box\varphi$, thus $M' \vDash_{x'} \Box\Diamond\neg\varphi$. So, using the same argument as above we get that $p \to \Diamond\neg\varphi \in$ KMcK.

The singleton reflexive frame $(\{0\}, \{\langle 0, 0\rangle\})$ is a frame for KMcK. Put a model on this frame which has p holding at 0. So this model must then satisfy $\Diamond\varphi$ and $\Diamond\neg\varphi$, so φ and $\neg\varphi$ hold at 0—a contradiction. Thus, $M' \vDash_{x'} \Diamond\Box\varphi$.

We are left with the requirement to show that KMcK $\nsubseteq$ S5 but this is straightforward for consider the model $M = (X, R, v)$, $X = \{0, 1\}$, $R = X \times X$, and $v(p) = \{0\}$. Then $M \vDash_0 \Box\Diamond p$ and $M \nvDash_0 \Diamond\Box p$. □

Theorem 9.14. *If L exhibits all consistencies in some index $i \in Idx$ and if $f : F^L_\kappa \longrightarrow F^L_\lambda$ is an isomorphism then there is a $g : \lambda \longrightarrow \mathcal{S}(\kappa)$ such that $f = g_+$, i.e., every isomorphism between the canonical frames for such logics is standard.*

Proof. Let $p \in \kappa$ be a propositional variable. Let $x \in X^L_\kappa$ be a maximal consistent extension of

$$\{\Box_i p\} \cup \{\Diamond_i\varphi \mid p \wedge \varphi \in \mathcal{S}(\kappa) \text{ is } L\text{-consistent}\}.$$

$\underline{\langle x, y\rangle \in R_{\kappa i} \Longleftrightarrow y \in \|p\|.}$

($\Longrightarrow$) Suppose that $\langle x, y\rangle \in R_{\kappa i}$. Since $\Box_i p \in x$, $p \in y$.

($\Longleftarrow$) Let $y \in \|p\|$. We want to show that

$$(\forall \varphi \in y)\left[\Diamond_i \varphi \in x\right].$$

But $y \in \|p\|$, so $p \wedge \varphi$ is consistent for all $\varphi \in y$. Thus for all $\varphi \in y$, $\Diamond_i \varphi \in x$.

Thus

$$\begin{aligned} f\left[\|p\|\right] &= f\left[\{y \mid \langle x, y\rangle \in R_{\kappa i}\}\right] \\ &= \{y \mid \langle f(x), y\rangle \in R_{\lambda i}\} \\ &= \{y \mid \{\varphi \mid \Box_i \varphi \in f(x)\} \subseteq y\} \\ &= \|\{\varphi \mid \Box_i \varphi \in f(x)\}\|. \end{aligned}$$

Now, let x' be a maximal consistent extension of

$$\{\Box_i \neg p\} \cup \{\Diamond_i \varphi \in \mathcal{S}(\kappa) \mid \neg p \wedge \varphi \text{ is } L\text{-consistent}\}$$

(which is consistent via a simple substitution argument). Again, we have that $\langle x', y\rangle \in R_{\kappa i} \Longleftrightarrow y \in \|\neg p\|$. Thus $f\left[\|\neg p\|\right] = \|\{\varphi \mid \Box_i \varphi \in f(x')\}\|$. But $\|p\| \cup \|\neg p\| = X_\kappa$, so $f\left[\|\neg p\|\right] \cup f\left[\|p\|\right] = X_\lambda$. Also, $f\left[\|\neg p\|\right] \cap f\left[\|p\|\right] = \varnothing$. Now, both $f\left[\|\neg p\|\right]$ and $f\left[\|p\|\right]$ are the intersection of clopen sets and so closed. Since they are the complements of each other, they must also be clopen. Proposition 2.35 tells us that the only clopen sets in a Stone space are those induced by elements of the underlying algebra (formulae), so $f\left[\|p\|\right] = \|\varphi\|$ for some formula φ.

In this way we see that f can be thought of as a map from the algebra $\underline{A}^L_\kappa$ to the algebra $\underline{A}^L_\lambda$. Since f^{-1} also induces such a map (which is in the reverse direction and undoes the effects of f), we see that f is essentially an $\mathcal{S}$-algebra isomorphism. Setting $g(p)$ to be the least (in the lexicographic sense) φ such that $f^{-1}\left[\|p\|\right] = \|\varphi\|$ gives us our desired g. □

Remark 9.15. It is impossible to sharpen this result so that g can actually be taken to be a map from λ to κ since there are many maps which accomplish such $\mathcal{S}$-algebra isomorphisms without being of this form. Trivially, consider $g : \lambda \longrightarrow \mathcal{S}(\lambda)$ given by $g(p) = \neg p$. Or, for a more involved example, consider g given by:

$$g(p) = \begin{cases} (p_1 \wedge p_2) \vee (\neg p_1 \wedge \neg p_2) & \text{if } p = p_1 \\ p & \text{if } p \neq p_1. \end{cases}$$

It is easy to check that $g^2 = \mathrm{id}$. Thus g_+ is an automorphism of F^L_λ, but it is not of the form $g : \lambda \longrightarrow \lambda$.

Question 9.16. Can this result be extended so that it encompasses arbitrary frame homomorphisms? Not by the proof given here since if $f : F_\lambda^L \longrightarrow F_\kappa^L$ is a frame homomorphism and $f^{-1}[\{x\}]$ is infinite then

$$f^{-1}[R(x)] = \{z \mid (\exists x')[f(x') = x \text{ and } \langle x', z\rangle \in R_i]\},$$

and this need not be a closed set.

Corollary 9.17. *For logics L which exhibit all consistencies, $F_\lambda^L \not\cong F_\kappa^L$ for all cardinals λ, $\max(\{\omega, \mathrm{card}(Idx)\}) \leqslant \lambda < \kappa$.*

Proof. Any such isomorphism would induce a one-one map $g : \kappa \longrightarrow \mathcal{S}(\lambda)$ which would contradict $\kappa > \lambda$. □

We see that this result also limits the possibilities for automorphisms of the canonical frame itself.

Corollary 9.18. *For logics L which exhibit all consistencies, F_λ^L has no non-standard automorphisms, for all cardinals $\lambda \geqslant \max(\{\omega, \mathrm{card}(Idx)\})$*

However, there is a distinction between the approach we have used here and that of the previous section:

Example 9.19. S5 does not exhibit all consistencies and F_κ^{S5} does have non-standard automorphisms. This is because it has 2^κ many equivalence classes of each allowable size which means that there are 2^{2^κ} many automorphisms, a number far greater than 2^κ, the number of standard automorphisms.

This is suggestive:

Conjecture 9.20. L exhibits all consistencies iff F_κ^L has no non-standard automorphisms.

9.5 Logics of Bounded Alternative and Granular Maps

This section highlights a type of isomorphism which we call 'granular.' These maps capture the idea that we could use $2^\lambda = 2^\kappa$ to guarantee a bijection between $\mathcal{P}(\lambda)$ and $\mathcal{P}(\kappa)$ which we then use as a base on which to build an isomorphism between F_κ^L and F_λ^L.

Our development will rely on the machinery of ultrafilter maps introduced in Chapter 8. While that chapter concerned itself mostly with ultrafilter embeddings constructed in a rather restricted way, we will allow a wider class of ultrafilter maps which we will then specialise in a new way.

Recall that $\Omega_{P,Q} := \{\mu \mid \mu : Q \longrightarrow \mathcal{S}(P)\}$. We now set:

Definition 9.21. $\Omega^0_{P,Q} := \{\mu \mid \mu : Q \longrightarrow \mathcal{S}_0(P)\}$

The ultrafilter embeddings, $h_{\mathcal{U}}$ for $\mathcal{U}$ an ultrafilter over $\Omega_{P,Q}$, which were introduced in Chapter 8 generalise standard frame homomorphisms and like standard frame homomorphisms there are maps which we call *stable* (that result from functions which send propositional letters to $\Box_i$-free or degree zero formulae) and maps which we call *unstable*. More formally:

Definition 9.22. A standard frame homomorphism $f : F^L_P \longrightarrow F^L_Q$ is *stable* iff for each $q \in Q$, $f^{-1}[\|q\|] = \|\varphi\|$ for some intensional operator free φ, and *unstable* otherwise.

An ultrafilter $\mathcal{U}$ on $\Omega_{P,Q}$ is *stable* iff $\Omega^0_{P,Q} \in \mathcal{U}$, and *unstable* otherwise. In the case of $\mathcal{U}$ being stable, $h_{\mathcal{U}}$ is also called *stable*.

From now on, fix $\mathcal{U}$ as a stable ultrafilter and consider $h_{\mathcal{U}} : X^L_P \longrightarrow X^L_Q$.

Since stable maps do not interfere with the modal depth of formulae we can analyse their action by seeing how they behave with respect to pairs of points which are equivalent up to formulae of some bounded modal depth.

Definition 9.23. For any set Q of propositional variables, and $n \in \omega$, define $\sim_n$, an equivalence relation on X^L_Q, by

$$x \sim_n y \iff x \cap \mathcal{S}_n(Q) = y \cap \mathcal{S}_n(Q).$$

Proposition 9.24.

$$(\forall n \in \omega)\left(\forall x, y \in X^L_P\right)[x \sim_n y \Longrightarrow h_{\mathcal{U}}(x) \sim_n h_{\mathcal{U}}(y)].$$

Proof. Suppose that $x \sim_n y$. Let $\varphi \in \mathcal{S}_n(Q)$, then

$$\begin{aligned}
\varphi \in h_{\mathcal{U}}(x) &\iff \{\mu \mid \mu(\varphi) \in x\} \in \mathcal{U} \\
&\iff \left\{\mu \in \Omega^0_{P,Q} \mid \mu(\varphi) \in x\right\} \in \mathcal{U} \\
&\iff \left\{\mu \in \Omega^0_{P,Q} \mid \mu(\varphi) \in y\right\} \in \mathcal{U} \text{ since } \mu(\varphi) \in \mathcal{S}_n(Q) \\
&\iff \varphi \in h_{\mathcal{U}}(y).
\end{aligned}$$

□

Because $h_{\mathcal{U}}$ is a stable map we can be assured that it will be kind to the number of i-successors of a point and this will be the crux of our proof that a logic need only be of bounded alternative for F^L_κ to have non-standard automorphisms.

Unfortunately to achieve our goals we cannot simply talk about the number of i-successors a point has, but we must modify the concept to talk about the number of $\mathcal{S}_n(Q)$-distinct i-successors that the point has.

Definition 9.25. We say that a point $x \in X_P^L$ has exactly (at most) $m/{\sim_n}$ i-successors, for $i \in \mathit{Idx}$ iff $R_{P_i}^L(x) / \sim_n$ has size exactly (at most) m.

Proposition 9.26. *A point $x \in X_P^L$ has at most $m/{\sim_n}$ i-successors, for $i \in \mathit{Idx}$ if and only if $\mathrm{Alt}_m^i(\overline{\varphi}) \in x$ for all $\overline{\varphi} \in \mathcal{S}_n(P)$.*

Proof. We prove the if and only if parts separately.

($\Longrightarrow$) Suppose that $\mathrm{Alt}_m^i(\overline{\varphi}) \notin x$ for some $\overline{\varphi} \in \mathcal{S}_n(P)$. Thus $\Diamond_i\varphi_0 \wedge \cdots \wedge \Diamond_i\varphi_m \in x$ and $\bigvee_{j<k\leqslant m} \Diamond_i(\varphi_j \wedge \varphi_k) \notin x$. Let $y_0, \ldots, y_m$ be in $R_{P_i}^L(x)$ so that $\varphi_k \in y_k$ for all $k \leqslant m$.

$j < k \Longrightarrow y_j \not\sim_n y_k$.

Suppose that $j < k$, then $\Diamond_i(\varphi_j \wedge \varphi_k) \notin x$ and so $\varphi_j \wedge \varphi_k \notin y_j \cup y_k$. But $\varphi_j \in y_j$ and $\varphi_k \in y_k$, so $\varphi_k \notin y_j$ and $\varphi_j \notin y_k$ and we conclude that $y_j \not\sim_n y_k$.

Thus $R_{P_i}^L(x) / \sim_n$ has size $> m$.

($\Longleftarrow$) Suppose that $R_{P_i}^L(x) / \sim_n$ has size greater than m. So there are $y_0, \ldots, y_m$ in this set which are pairwise $\sim_n$-inequivalent and we get that there are formulae $\varphi_0, \ldots, \varphi_m \in \mathcal{S}_n(Q)$ which form a disjoint cover of the whole space with a unique cover element to each y_j. In particular we have:

1. $(\forall j < k \leqslant m)\,[\vdash_L \neg(\varphi_j \wedge \varphi_k)]$,
2. $\vdash_L \bigvee_{j\leqslant m} \varphi_j$, and
3. $(\forall j \leqslant m)\,[\varphi_j \in y_j]$.

Thus $\Diamond_i\varphi_0 \wedge \cdots \wedge \Diamond_i\varphi_m \in x$. But by 1 (above) we have that for all $j < k \leqslant m$, $\neg\Diamond_i(\varphi_j \wedge \varphi_k) \in x$, establishing that $\mathrm{Alt}_m^i(\overline{\varphi})$ is not in x.

□

Corollary 9.27. *Let $i \in \mathit{Idx}$. If $x \in X_P^L$ has exactly $m/{\sim_n}$ i-successors and $x \sim_{n+1} y$ then y has $m/{\sim_n}$ i-successors.*

Proof. Assume that there is a case where x has exactly $m/{\sim_n}$ i-successors, and y has exactly $k/{\sim_n}$ i-successors, for $k > m$. Thus

1. $\mathrm{Alt}_m^i(\overline{\varphi}) \in x$ for all $\overline{\varphi} \in \mathcal{S}_n(P)$, and
2. $\mathrm{Alt}_m^i(\overline{\varphi}) \notin y$ for some $\overline{\varphi} \in \mathcal{S}_n(P)$.

But we can clearly see that $\mathrm{Alt}_m^i(\overline{\varphi})$ is in $\mathcal{S}_{n+1}(P)$ allowing us to conclude that $x \not\sim_{n+1} y$. □

Having established the properties of the concept of $m/{\sim_n}$ i-successors we can now go on to prove that stable maps leave these numbers unchanged.

Proposition 9.28. *Let $i \in Idx$. If $x \in X_P^L$ has exactly $m/{\sim_n}$ i-successors then $h_{\mathcal{U}}(x)$ has at most $m/{\sim_n}$ i-successors.*

Proof. Suppose that x has $m/{\sim_n}$ i-successors. Then we have that

$$(\forall \overline{\varphi} \in \mathcal{S}_n(P))\left[\mathrm{Alt}_m^i(\overline{\varphi}) \in x\right]$$

and so we will have the result if we can show:
$(\forall \overline{\varphi} \in \mathcal{S}_n(Q))\left[\mathrm{Alt}_m^i(\overline{\varphi}) \in h_{\mathcal{U}}(x)\right]$.

Let $\overline{\varphi} \in \mathcal{S}_n(Q)$. For $\mu \in \Omega_{P,Q}^0$,

$$\mu\left(\mathrm{Alt}_m^i(\overline{\varphi})\right) = \mathrm{Alt}_m^i(\mu(\overline{\varphi})) \in x$$

since $\deg \mu(\varphi_j) = \deg \varphi_j \leqslant n$. But we require that $\Omega_{P,Q}^0 \in \mathcal{U}$ giving $\mathrm{Alt}_m^i(\overline{\varphi}) \in h_{\mathcal{U}}(x)$.

□

With the $m/{\sim_n}$ i-successor preliminaries out of the way we can show that for L a logic of bounded alternative (in all indices), the action of $h_{\mathcal{U}}$ is essentially determined by how it acts on sets of propositional variables.

Definition 9.29. For P a set of propositional variables and $a \subseteq P$, let z_a^P be the propositional calculus closure of the set $a \cup \{\neg p \mid p \in P - a\}$ within $\mathcal{S}_0(P)$.

We will drop the superscript P in z_a^P since throughout the rest of this chapter there will be no ambiguity introduced by such a move.

Definition 9.30. A map $f : X_P^L \longrightarrow X_Q^L$ is called *granular* iff, for all $a \subseteq P$, there is a $b \subseteq Q$ so that $f[\|z_a\|] \subseteq \|z_b\|$. In the case of f a granular map define $G(f) : \mathcal{P}(P) \longrightarrow \mathcal{P}(Q)$, the *granularity of f* by

$$G(f)(a) = \left\{p \in Q \mid p \in \bigcap f[\|z_a\|]\right\}.$$

An immediate consequence of this definition is that if $f : X_P^L \longrightarrow X_Q^L$ is granular then $f[\|z_a\|] \subseteq \left\|z_{G(f)(a)}\right\|$.

Whereas standard maps have, at their very essence, a mapping of propositional variables, granular maps are less constrained and only have maps between sets of propositional letters. Clearly the standard maps induced by maps of the form $q \mapsto \pm p$ are granular maps.

Proposition 9.31. *If L is a logic of bounded alternative (in all indices) and if $h_{\mathcal{U}}$ ($\mathcal{U}$ a stable ultrafilter) is a granular map with $G(h_{\mathcal{U}}) : \mathcal{P}(P) \longrightarrow \mathcal{P}(Q)$ an injection then $h_{\mathcal{U}} : X_P^L \longrightarrow X_Q^L$ is also an injection. Moreover, in this case we have that*

$$(\forall n \in \omega) \left(\forall x, y \in X_P^L\right) [h_{\mathcal{U}}(x) \sim_n h_{\mathcal{U}}(y) \Longrightarrow x \sim_n y].$$

Proof. We will prove the second conclusion of the proposition by induction on n and this will give us the first conclusion. For the base case, note that if $x \not\sim_0 y$ then there are distinct $a, b \in \mathcal{P}(P)$ with $x \in \|z_a\|$ and $y \in \|z_b\|$, thus by $G(h_{\mathcal{U}})$ one-one, $c := G(h_{\mathcal{U}})(a)$ and $d := G(h_{\mathcal{U}})(b)$ are distinct with $h_{\mathcal{U}}(x) \in \|z_c\|$ and $h_{\mathcal{U}}(y) \in \|z_d\|$, allowing us to conclude that $h_{\mathcal{U}}(x) \not\sim_0 h_{\mathcal{U}}(y)$.

Our inductive hypothesis is that the result holds for n and we now try to prove the result for $n+1$:

Suppose that $h_{\mathcal{U}}(x) \sim_{n+1} h_{\mathcal{U}}(y)$. By Corollary 9.27 we have that $h_{\mathcal{U}}(x)$ and $h_{\mathcal{U}}(y)$ have exactly the same number of $\sim_n$ i-successors modulo $\sim_n$ for each $i \in \mathit{Idx}$. We now show that, for each $i \in \mathit{Idx}$, x and y then have the same number of $R_{P_i}^L$-successors up to $\sim_n$-equivalence:
$(\forall i \in \mathit{Idx}) \left(\forall x' \in R_{P_i}^L(x)\right) \left(\exists y' \in R_{P_i}^L(y)\right) [x' \sim_n y']$.

> Let $x' \in R_{P_i}^L(x)$ and assume that $x' \not\sim_n y'$ for all $y' \in R_{P_i}^L(y)$. From the inductive hypothesis we must then conclude that $h_{\mathcal{U}}(x') \not\sim_n h_{\mathcal{U}}(y')$ for all $y' \in R_{P_i}^L(y)$. Theorem 8.13 tells us that $h_{\mathcal{U}}(x')$ is distinct from all the $R_{Q_i}^L$ successors of $h_{\mathcal{U}}(y)$, and this can be witnessed by some $\varphi \in \mathcal{S}_n(Q)$ ($\varphi \in h_{\mathcal{U}}(x')$, $\neg\varphi \in h_{\mathcal{U}}(y')$ for all $y' \in R_{P_i}^L(y)$). Using the fact that $h_{\mathcal{U}}$ is a frame homomorphism again we get that $\Box_i \neg\varphi \in h_{\mathcal{U}}(y)$. But $\varphi \in h_{\mathcal{U}}(x')$, so $\Diamond_i \varphi \in h_{\mathcal{U}}(x)$, contradicting $h_{\mathcal{U}}(x) \sim_{n+1} h_{\mathcal{U}}(y)$.

Let $\varphi \in \mathcal{S}_n(P)$, we want to show that, for each $i \in \mathit{Idx}$,

$$\Diamond_i \varphi \in x \Longleftrightarrow \Diamond_i \varphi \in y$$

which will establish that $x \sim_{n+1} y$. So suppose that $i \in \mathit{Idx}$, then

$$\begin{aligned}
\Diamond_i \varphi \in x &\Longrightarrow \left(\exists x' \in R_{P_i}^L(x)\right) [\varphi \in x'] \\
&\Longrightarrow \left(\exists x' \in R_{P_i}^L(x),\ y' \in R_{P_i}^L(y)\right) [x' \sim_n y' \text{ and } \varphi \in x'] \\
&\Longrightarrow \left(\exists y' \in R_{P_i}^L(y)\right) [\varphi \in y'] \\
&\Longrightarrow \Diamond_i \varphi \in y.
\end{aligned}$$

The converse follows by symmetry. □

Recall the comment that granular frame homomorphisms are essentially determined by their granularity. This is basically due to the frame homomorphism theorem (a.k.a. the p-morphism theorem). Take for example this result.

Lemma 9.32. *Let L be a logic*[4] *and $h_1 : F_P^L \longrightarrow F_Q^L$, $h_2 : F_Q^L \longrightarrow F_P^L$ be frame homomorphisms so that $G(h_2) \circ G(h_1) = \mathrm{id}$. Then $h_1 \circ h_2 = \mathrm{id}$.*

Proof. This result is an immediate consequence of the frame homomorphism theorem (see e.g., [13, p. 98]). We have that $h_2 \circ h_1$ is a frame homomorphism and $h_2 \circ h_1$ is a mapping between canonical *models* which leaves the intensions of propositions fixed. In particular:
$\underline{(\forall x \in X_P^L)\,(\forall p \in P)\,[p \in x \Longleftrightarrow p \in h_2 \circ h_1(x)]}$

We will prove the *if* and *only if* parts side by side. Let $x \in X_P^L$ and suppose that $p \in x$ ($\neg p \in x$). Let $a \subseteq P$ be such that $x \in \|z_a\|$. Then

$$\begin{aligned} h_2 \circ h_1(x) &\in h_2 \circ h_1 [\|z_a\|] \\ &\subseteq h_2 \left[\left\|z_{G(h_1)(a)}\right\|\right] \\ &\subseteq \left\|z_{G(h_2)(G(h_1)(a))}\right\| \\ &= \|z_a\| \end{aligned}$$

and so $p \in h_2 \circ h_1(x)$ ($\neg p \in h_2 \circ h_1(x)$).

From this the frame homomorphism theorem allows us to conclude that

$$(\forall \varphi \in \mathcal{S}(P))\left[M_P^L \vDash_x \varphi \Longleftrightarrow M_P^L \vDash_{h_2 \circ h_1(x)} \varphi\right],$$

(where M_P^L is the canonical model over F_P^L) giving our result. □

We next note that any type of granularity can give rise to a map between canonical frames.

Lemma 9.33. *For $f : \mathcal{P}(P) \longrightarrow \mathcal{P}(Q)$ there is a stable $\mathcal{U}$ such that $G(h_{\mathcal{U}}) = f$.*

Proof. We will work within the Stone space $(Z, \mathcal{T})$ where $Z = \{z_a^P \mid a \subseteq P\}$, and $\mathcal{T}$ is the topology with clopen basis

$$\{\|\varphi\|_0 \mid \varphi \in \mathcal{S}_0(P)\}, \ (\|\varphi\|_0 := \{z \in Z \mid \varphi \in z\}).$$

We now need to construct the ultrafilter $\mathcal{U}$ and we will show that the natural definition does indeed work: For $\Delta \subseteq \mathcal{P}(P)$ finite, let U_Δ be the set

$$\{\mu \in \Omega^0_{P,Q} \mid (\forall a \in \Delta)\,(\forall q \in Q)\,[\mu(q) \in z_a \Longleftrightarrow q \in f(a)]\}.$$

Clearly $U_{\Delta_0} \cap U_{\Delta_1} = U_{\Delta_0 \cup \Delta_1}$ so to meet our goal of showing that the natural

$$\nabla := \{U_\Delta \mid \Delta \subseteq \mathcal{P}(P) \text{ is finite}\}$$

[4] *Not necessarily of bounded alternative.*

is a basis for an ultrafilter $\mathcal{U}$ we must show that $U_\Delta \neq \varnothing$ for all finite $\Delta \subseteq \mathcal{P}(P)$. So let $\Delta \subseteq \mathcal{P}(P)$ be finite. We must define a $\mu : Q \longrightarrow \mathcal{S}_0(P)$. Take an arbitrary $q \in Q$. Say

$$\begin{aligned} q &\in f(a_0) \cap \cdots \cap f(a_{m-1}), \\ q &\notin f(b_0) \cap \cdots \cap f(b_{n-1}), \text{ and} \\ \Delta &= \{a_j, b_k \mid j < m, k < n\}. \end{aligned}$$

We take it that these $\{a_j, b_k\}$ were chosen so that they are all distinct and so give rise to distinct points z_{a_j} $j < m$, z_{b_k} $k < n$. Since $\mathcal{T}$ is a Stone topology we can find basic open $\|\varphi\|_0$, for some $\varphi \in \mathcal{S}_0(P)$, so that $z_{a_j} \in \|\varphi\|_0$ for $j < m$ and $z_{b_k} \notin \|\varphi\|_0$ for $k < n$. Choose $\mu(q)$ to be this φ. Now suppose that $\mu(q)$ is so defined for all $q \in Q$. We must check that this meets our requirement:
$\underline{(\forall a \in \Delta)\,(\forall q \in Q)\,[\mu(q) \in z_a \Longleftrightarrow q \in f(a)]}$.

Let $a \in \Delta$, $q \in Q$, and $\{a_j, b_k \mid j < m, k < n\}$ be as before.

($\Longrightarrow$) Suppose that $q \notin f(a)$. Then $a = b_k$ for some $k < n$. Thus $z_{b_k} \notin \|\varphi\|_0$ giving $\mu(q) = \varphi \notin z_{b_k} = z_a$.

($\Longleftarrow$) Suppose that $q \in f(a)$. Thus $a = a_j$ for some $j < m$ and so $z_{a_j} \in \|\varphi\|_0$ giving $\mu(q) = \varphi \in z_{a_j} = z_a$.

Thus $\mu \in U_\Delta$ showing that $U_\Delta \neq \varnothing$.

Having completed this verification we can now proceed to show that $\mathcal{U}$, any ultrafilter which extends ∇, gives rise to a granular $h_\mathcal{U}$ satisfying $G(h_\mathcal{U}) = f$. In particular we show:
$\underline{(\forall a \in \mathcal{P}(P))\,[h_\mathcal{U}[\|z_a\|] \subseteq \|z_{f(a)}\|]}$.

Let $a \in \mathcal{P}(P)$, and $x \in \|z_a\|$, so $z_a \subseteq x$.

Let $q \in f(a)$. For all $\mu \in U_{\{a\}}$, $\mu(q) \in z_a \subseteq x$, so $q \in h_\mathcal{U}(x)$.

Let $q \notin f(a)$. For all $\mu \in U_{\{a\}}$, $\neg\mu(q) \in z_a \subseteq x$, so $\neg q \in h_\mathcal{U}(x)$, as desired.

□

We are now able to tie all these results together into theorems that are able to address the main points of this chapter.

Theorem 9.34. *In the event that L is a logic of bounded alternative, there are* $(2^\kappa)^{(2^\lambda)}$ *many frame homomorphisms from* F^L_λ *to* F^L_κ.

Proof. There are $(2^\kappa)^{(2^\lambda)}$ maps from $\mathcal{P}(\lambda)$ to $\mathcal{P}(\kappa)$, each one giving rise to a distinct $h_\mathcal{U}$ which must be a frame homomorphism. □

We note that there are λ^κ many standard maps between F^L_λ and F^L_κ (assuming that $\mathrm{card}\,(Idx) \leqslant \lambda$) and so in cases where $(2^\kappa)^{(2^\lambda)} > \lambda^\kappa$ (e.g., where $\kappa < \lambda$) there are non-standard frame homomorphisms between F^L_λ and F^L_κ.

We can now conclude the section and the chapter with the answer to the question about the number and type of automorphisms and isomorphisms between canonical frames for logics of bounded alternative.

Theorem 9.35. *For L a logic of bounded alternative (in all indices):*

1. *If $2^\lambda = 2^\kappa$ then $F^L_\lambda \cong F^L_\kappa$.*
2. *The frame F^L_λ has 2^{2^λ} many automorphisms, some of which must be non-standard.*

Proof. For 1: Suppose that $2^\lambda = 2^\kappa$. Then there is a bijection $f : \mathcal{P}(\lambda) \longrightarrow \mathcal{P}(\kappa)$. By Lemma 9.33 there are $h_\mathcal{U} : F^L_\lambda \longrightarrow F^L_\kappa$ and $h_{\mathcal{U}^{-1}} : F^L_\kappa \longrightarrow F^L_\lambda$ so that $G(h_\mathcal{U}) = f$ and $G(h_{\mathcal{U}^{-1}}) = f^{-1}$, and by Lemma 9.32, $h_{\mathcal{U}^{-1}} \circ h_\mathcal{U} = \mathrm{id}$ and $h_\mathcal{U} \circ h_{\mathcal{U}^{-1}} = \mathrm{id}$. Thus $h_\mathcal{U}$ is an isomorphism between F^L_λ and F^L_κ.

For 2: Each bijection $f : \mathcal{P}(\lambda) \longrightarrow \mathcal{P}(\lambda)$ corresponds to a distinct automorphism. There are 2^{2^λ} many such bijections and hence 2^{2^λ} many such automorphisms. There are only 2^λ many maps from λ to $\mathcal{S}(\lambda)$, hence only 2^λ many standard automorphisms. □

9.6 Conclusion

We have seen that there are a number of interesting questions which arise when we look at the isomorphisms over and between canonical structures, and we have highlighted the classes of standard and non-standard maps. We have looked at logically simple systems like S5 and have seen that while we can make many interesting observations, even here some difficult set-theoretic questions can arise.

By using the technique of ultrafilter embeddings we have investigated the isomorphisms and automorphisms over logics of bounded alternative and have seen that this is one realm in which we can tell a relatively complete story.

We have only scratched the surface of the questions that arise from the inquiry "Does F^L_κ have any non-standard automorphisms?" It is the author's hope that this work will stimulate interest in this area and that it will encourage researchers to refine our understanding of canonical frame automorphisms (or even frame homomorphisms).

Chapter 10

Conclusions

This thesis has looked at the big canonicity problems in modern intensional logic from a number of different points of view. The approaches associated with these points of view have:

1. Led to new results which may well lie on the road to solutions to these problems.
2. Had a common thread so that by highlighting and homing in on this thread we can try to acquire a better understanding of underlying mechanisms.
3. Led us to new and interesting questions which are deserving of attention in future work.

10.1 Points of View

The canonicity questions were attacked from three points of view. The first was in terms of algebraic semantics. Even though these semantics were not able to capture the definitions of canonicity, they were a constant reminder of where power set boolean algebras were important (see, e.g., Theorem 6.8 which worked for any boolean algebra $\underline{B}$).

The second point of view was that of relational semantics. While this is also a traditional means of analysing intensional logics, it was necessary because it allowed us to pose and answer questions about maps between Stone spaces—Chapters 8 and 9. When we tried to do without the direct use of relational semantics in Chapter 7, we got a better idea of what was going on, and an appreciation of how important an accessibility relation can be.

The final point of view was that of neighborhood semantics. We have seen how this approach forces us to think algebraically yet with reference to the underlying powerset behaviour. We were able to get canonicity results even though the relational semantics was not there to help. This did highlight the fact that, as with completeness, neighborhood semantics properly sits between algebraic and relational semantics—giving us a useful, but not perfect tool, in approaching the long standing canonicity questions, which, after all, were about relational canonicity. It also gave us new questions on canonical existence to ponder.

10.2 Specific Questions Answered

By taking these three approaches we have been able to make the following discoveries:

1. All non-iterative logics are neighborhood canonical.
2. All even logics which have the finite model property are neighborhood canonical. This means that all uniform normal modal logics are neighborhood canonical and so, through the McKinsey logic, we have that neighborhood canonicity is more general than relational canonicity.
3. All logics below S5 do not have non-standard isomorphisms between F^L_ω and $F^L_{\omega_1}$, yet they may have non-standard automorphisms of these structures.
4. All logics which admit all consistencies cannot have non-standard isomorphisms or automorphisms.
5. All logics of bounded alternative have many non-standard isomorphisms and automorphisms.

10.3 Unifying Thread: Ultrapowers

Throughout this work we have blatantly and frequently made use of ultrafilters, yet it might not be immediately clear that we have, in some form, always been using ultrapowers. In Chapters 5 and 6 our use of ultrapowers was in our expansion of the Lindenbaum algebra into something that included the canonical frame. The use here was blatant because we openly used ultrapowers. Note here that each element of the canonical algebraic frame, that is, each subset of the canonical space, was represented as some element of the ultrapower.

In Chapter 7 we introduced our ultrafilter semantics which purported to represent each subset of the canonical Stone space as a formula in a suitably large language. Even though we did not make it explicit, each of the formulae in the suitably large language could, again, be thought of as elements of an ultrapower of the original language. After all, $||\varphi||_{\mathcal{U}}$ really was just the inverse image of the element $[\widetilde{\varphi}]_{\sim_{\mathcal{U}}}$ under the Łos function, where $\widetilde{\varphi} : \Omega \longrightarrow \mathcal{S}(P)$ is defined by $\widetilde{\varphi}(\mu) = \mu(\varphi)$. Chapter 7 showed the remarkable ability of ultrafilter semantics—and hence the ultrapowers hanging around in the background—to recreate a significant subset of known canonicity results.

In Chapter 8 and its dependent, Chapter 9, we also made use of ultrapowers in that each $h_{\mathcal{U}}(x)$ is the projection of a *point* in the ultrapower of the canonical frame back down into a canonical frame of different cardinality.

10.4 New Questions

In each chapter the underlying ultrapowers did not drive our analysis so there is an opportunity to determine the exact role that ultrapowers can play in canonicity questions.

Also, our analysis of ultrafilter semantics in Chapter 7 may not exhaustively cover all the known cases of relational canonicity so it is important to determine whether this technique can be extended or if it is inadequate for the task, and if so, why.

Even given the level of attack presented in Appendix A, the canonicity question for the logic EK4 remains open. This lack of success puts the neighborhood completeness of this logic into doubt. Any form of completeness result would provide invaluable information on how iterativity affects semantic truth.

While we now have some idea of which logics will admit non-standard isomorphisms, we still have a large class of logics which are not covered by our theorems. Is it possible that the existence of non-standard isomorphisms is related to the realisation of all consistencies as conjectured in Chapter 9? What about non-standard frame homomorphisms? Apart from simple logics of bounded alternative, we have been able to say nothing here.

And finally, there are still the major problems of "Does elementarity imply canonicity?" and "Does canonicity in one cardinal imply canonicity in higher cardinals?" which this thesis has not been able to solve.

These are difficult problems! Hopefully the reader now has a few more ideas on how they could be tackled and an appreciation of the new approaches presented here.

Appendix A

EK4, a case study

A.1 Introduction

The results of Chapter 5 naturally suggest the question of what happens when we move to logics which are iterative: Which iterative logics are canonical and which are not? Of course, for the normal modal logics this question is well understood for a host of interesting iterative logics, but this is entirely because we have the accessibility relation to work with. In the region which is well removed from the logic K it is not clear how to proceed.

As we saw in Chapter 4 some classical iterative logics are quickly shown to be canonical and so complete, namely those logics which have axioms that do not mix propositional variables. An example is the simple iterative logic E4. This is trivially complete because it satisfies the criteria of BENTON's result, however 'Does the logic E4 have the finite frame property?' remains an open question (see WOLTER's paper [105]).

When thinking of iterative logics for which completeness is unknown, the simplest logic that comes to mind is the logic EK4—which is the classical mono-modal logic with axioms:

1. $\Box(p \to q) \to (\Box p \to \Box q)$,

2. $\Box p \to \Box\Box p$, and is closed under the rule

3. from $\varphi \leftrightarrow \psi$ infer $\Box\varphi \leftrightarrow \Box\psi$.

The following questions are currently unanswered:

Question A.1. Is the logic EK4 canonical?

Question A.2. Does the logic EK4 have the finite frame property?

Of course, in the light of our work in Chapter 6, a positive answer to the second question will give an immediate positive answer to the first. While this work is unable to answer these two questions, some progress has been made in understanding EK4 itself and this appendix is presented as a brief introduction to the behaviour of the system so that other researchers can take up the cause.

Throughout this appendix we will mostly work with neighborhood semantics rather than algebraic frames and our logics are mono-modal so we will suppress the subscripts on the $\Box$s and the Ns. Also we will take our set of propositional letters to be fixed as P which we take to be ω.

A.2 K Frames

We will start by looking at the underlying EK logic and all EK, and thus EK4, frames. Since the EK part of the logic is non-iterative it is only necessary to look at how the neighborhoods at each point behave:

If $(X, N) \models \mathsf{EK}$ then each $N(x)$ will be a set which, in essence, "satisfies" K all by itself. This motivates the following definition.

Definition A.3. Let $\underline{A}$ be a boolean algebra. A *neighborhood set* is any subset of A. We say that the neighborhood set V is K-*like* iff

$$(\forall a, b \in A)\,[\neg a \vee b \in V \text{ and } a \in V \implies b \in V]\,.$$

Now, to make our analysis a little easier let us introduce a concept based on simple graph theory.[1]

Definition A.4. For $\underline{A}$ a boolean algebra define graph $(\underline{A}) = (A, E(\underline{A}))$ to be *the* K-*graph of* $\underline{A}$ with an edge set defined as follows:

$$E(\underline{A}) = \{\{a, a'\} \mid a \neq a', \neg a \leqslant a'\}\,.$$

Now, each graph is divided up into its connected components:

Definition A.5. Let $\mathcal{G} = (G, E)$ be a graph, and let $g \in G$. then

$$\begin{aligned} [g] &= \{g' \in G \mid g' E^{\star} g\} \text{ where} \\ E^{\star} &= \{(g, g') \mid (\exists n \in \omega)\,(\exists g_0, \ldots, g_n \in G) \\ &\qquad [g = g_0 \text{ and } g' = g_n \text{ and } (\forall i < n)\,[\{g_i, g_{i+1}\} \in E]]\}. \end{aligned}$$

This is called the *g-component of* $\mathcal{G}$.

[1] We take BELA BOLLOBAS's book [8] to be our basic graph theory reference.

Lemma A.6. *Let $\underline{A}$ be a boolean algebra and V a* K*-like neighborhood set on $\underline{A}$. Let $\mathcal{G}$ be the subgraph of* graph $(\underline{A})$ *formed by restricting* graph $(\underline{A})$ *to V. Let $\langle a_0, \ldots, a_n \rangle$ be a path through $\mathcal{G}$, i.e., $(\forall i < n)\,[\neg a_i \leqslant a_{i+1}]$. Then*

$$(\forall b \in A)\,[a_0 \wedge \cdots \wedge a_n \leqslant b \leqslant a_n \Longrightarrow b \in V]\,.$$

Proof. We proceed by induction on n, the length of the path.
Base Cases: $n = 0$. Then $a_0 \leqslant b \leqslant a_0 \Longrightarrow b = a_0 \in V$.
Inductive Hypothesis: Assume that the result holds for paths of length n.
Inductive Step: Suppose that $\langle a_0, \ldots, a_n, a_{n+1} \rangle$ is a path through $\mathcal{G}$. Let $c = (a_0 \wedge \cdots \wedge a_n) \vee \neg a_{n+1}$. Thus

$$a_0 \wedge \cdots \wedge a_n \leqslant c \leqslant a_n \vee \neg a_{n+1} \leqslant a_n \vee a_n = a_n,$$

so $c \in V$. Now,

$$\neg c = \neg(a_0 \wedge \cdots \wedge a_n) \wedge a_{n+1} \leqslant a_{n+1} \in V,$$

thus $(\forall b \in A)\,[c \wedge a_{n+1} \leqslant b \leqslant a_{n+1} \Longrightarrow b \in V]$. (This follows because if $c \wedge a_{n+1} \leqslant b \leqslant a_{n+1}$ then $a_{n+1} \geqslant \neg c \vee b \geqslant \neg c \vee (c \wedge a_{n+1}) = (\neg c \vee c) \wedge (a_{n+1} \vee \neg c) = \top \wedge a_{n+1} = a_{n+1}$, so $\neg c \vee b \in V$ and $c \in V$, so $b \in V$ 'by' K.) But

$$\begin{aligned} c \wedge a_{n+1} &= ((a_0 \wedge \cdots \wedge a_n) \vee \neg a_{n+1}) \wedge a_{n+1} \\ &= (a_0 \wedge \cdots \wedge a_n \wedge a_{n+1}) \vee (\neg a_{n+1} \wedge a_{n+1}) \\ &= a_0 \wedge \cdots \wedge a_{n+1}. \end{aligned}$$

□

Theorem A.7. *Let $\underline{A}$ be a boolean algebra and V a* K*-like neighborhood set on $\underline{A}$. Let $\mathcal{G}$ be the subgraph formed by restricting* graph $(\underline{A})$ *to V. Let $[a]$ be a component of $\mathcal{G}$. Then*

$$(\forall n \in \omega)\,(\forall a_0, \ldots, a_n \in [a])\,(\forall b \in A)\,[a_0 \wedge \cdots \wedge a_n \leqslant b \leqslant a_0 \Longrightarrow b \in V]\,.$$

Proof. Let $n \in \omega$ and let $a_0, \ldots, a_n \leqslant b \leqslant a_0$. There must be a path $\langle a'_0, \ldots, a'_m \rangle$ through $[a]$ which visits each a_i and ends at a_0. Thus

$$a'_0 \wedge \ldots \wedge a'_m \leqslant a_0 \wedge \ldots \wedge a_n \leqslant b \leqslant a_0 = a'_m$$

so by Lemma A.6 $b \in V$. □

Thus we see that a K-like neighborhood set, and hence the neighborhood sets in an EK frame are "flower like" in the sense of Figure A.1.

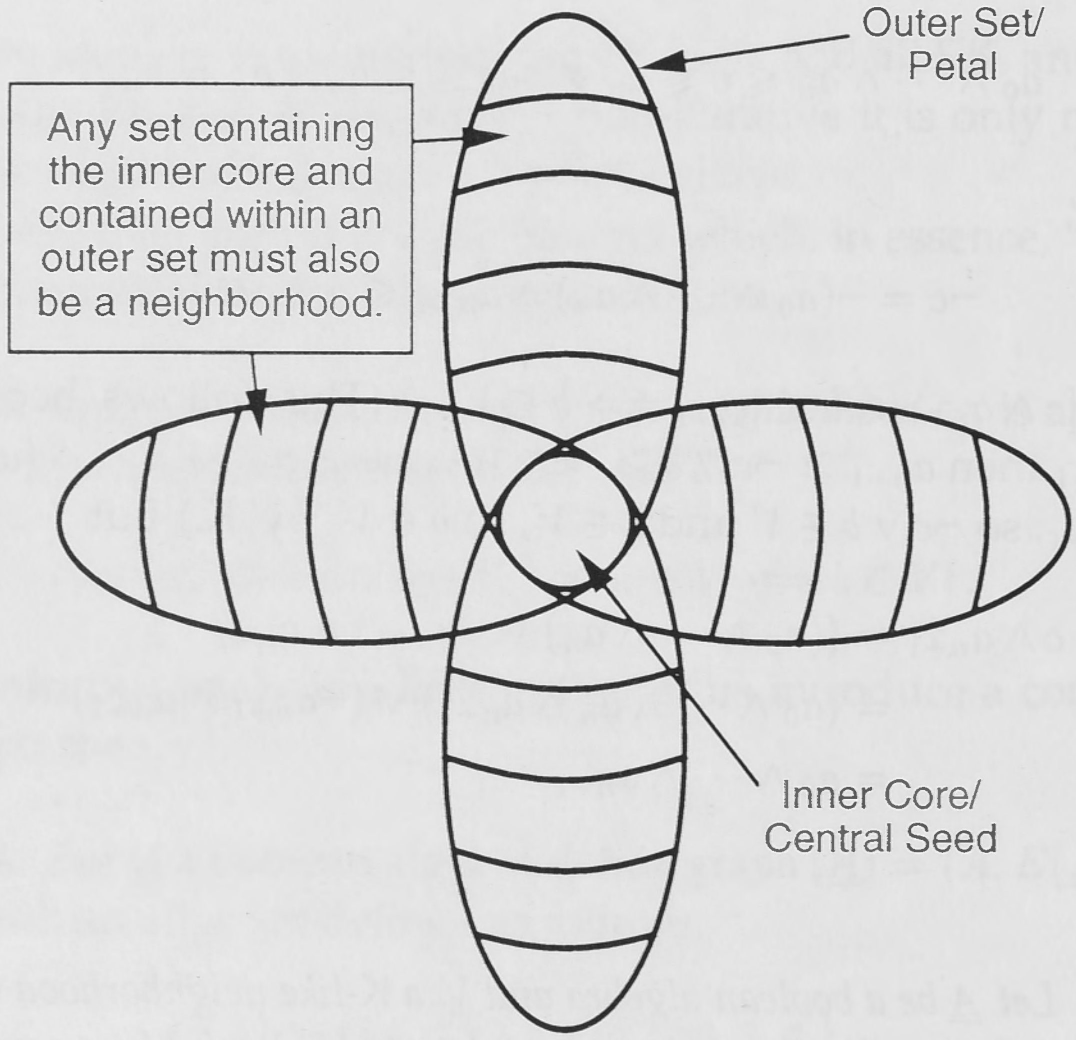

Figure A.1: A conceptualisation of a K-like neighborhood set.

The neighborhoods must include a central 'seed' but then they are free to include anything between that and the outer edge of one of the petals.

This whole result means that we can simplify our understanding of an EK-neighborhood set by homing in on that central kernel while keeping some outer petal in mind. In the same way that a normal K-neighborhood set is closed under intersection, we have that the members of the neighborhood that are contained in one of these 'outer sets' (petals) form a set that is closed under intersection. So we could construct an accessibility relation for each of the outer sets. But this would not help us deal with the canonicity of iterative axioms as there is no clear prescription for how to combine relations—something we do with the single relation of normal modal logic. [Suppose that we have a relation $R_{||\varphi||}$ based on some outer set $||\varphi||$ and another relation $R_{||\psi||}$ based on $||\psi||$. Will $R_{||\varphi||} \circ R_{||\psi||}$ have any semantic meaning? After all, this is no longer relative to a single outer set.]

Note that if we look at the worlds with $||\top|| = X \in N(x)$ in an EK4-frame, we have only one component as all $Y \subseteq X$ will satisfy $\neg Y \subseteq ||\top||$. This suggests that by restricting ourselves to worlds in $||\Box\top||$ we will, in some sense, get a generated subframe of (X, N).

To show that this is indeed the case we must first introduce two minor pieces of notation: For (X, N) a frame, let $||\Box\top|| = \{x \in X \mid X \in N(x)\}$. For $M = (X, N, v)$ a model and $\varphi \in \mathcal{S}(P)$ let

$$||\varphi||^M = \{x \in X \mid M \vDash_x \varphi\}.$$

Proposition A.8. *Let* $(X, N) \vDash$ EK4 *and let* v *be a valuation on* (X, N)*. Set* $N'(x) = \{Y \cap ||\Box\top|| \mid Y \in N(x)\}$*. Let* v' *be a valuation on* $(||\Box\top||, N')$ *defined by* $v'(p) = v(p) \cap ||\Box\top||$*. Let* $M = (X, N, v)$ *and let* $M' = (||\Box\top||, N', v')$*. Then*

$$(\forall\varphi \in \mathcal{S}(P)) \left[||\varphi||^M \cap ||\Box\top|| = ||\varphi||^{M'}\right].$$

Proof. This proposition is proved by induction on the complexity of φ. The base and boolean cases are immediate so we must show that for $\varphi \in \mathcal{S}(P)$, $||\Box\varphi||^M \cap ||\Box\top|| = ||\Box\varphi||^{M'}$:

$$\begin{aligned} ||\Box\varphi||^M \cap ||\Box\top|| &= ||\Box\top|| \cap \left\{x \in X \mid ||\varphi||^M \in N(x)\right\} \\ &= \left\{x \in ||\Box\top|| \mid ||\varphi||^M \in N(x)\right\} \\ &= \left\{x \in ||\Box\top|| \mid ||\varphi||^{M'} \in N(x)\right\} \\ &= ||\Box\varphi||^{M'}. \end{aligned}$$

The third equality is justified as follows:

($\subseteq$) Suppose that $x \in \|\Box\top\|$ and $\|\varphi\|^M \in N(x)$. Thus $\|\varphi\|^{M'} = \|\varphi\|^M \cap \|\Box\top\| \in N'(x)$ by definition.

($\supseteq$) Suppose that $x \in \|\Box\top\|$ and $\|\varphi\|^{M'} \in N'(x)$. Thus there exists a $Y \in N(x)$ such that

$$Y \cap \|\Box\top\| = \|\varphi\|^{M'} = \|\varphi\| \cap \|\Box\top\|.$$

Without loss of generality we can assume that $Y = v(q)$ for some propositional letter q not appearing in φ. Thus $x \in \|\Box q\|^M$ and so $q \wedge \Box\top \leftrightarrow \varphi \wedge \Box\top \in \mathcal{T}h(M)$.

Claim: $\Box\top \to (\Box q \leftrightarrow \Box\varphi) \in \mathcal{T}h(M)$.

Proof of claim:

We provide a derivation sequence:

1. $\top \leftrightarrow (q \wedge \Box\top \to q) \in Taut$.
2. $\Box\top \leftrightarrow (\Box(q \wedge \Box\top) \to \Box q) \in$ EK - applying K to 1.
3. $\top \leftrightarrow (q \to (\Box\top \to q \wedge \Box\top)) \in Taut$.
4. $\Box\top \to (\Box q \wedge \Box\Box\top \to \Box(q \wedge \Box\top)) \in$ EK - applying K to 3.
5. $\Box\top \to \Box\Box\top \in$ EK4.
6. $\Box\top \to (\Box q \to \Box(q \wedge \Box\top)) \in$ EK4 - 4,5 and PC.[2]
7. $\Box\top \to (\Box\varphi \leftrightarrow \Box(\varphi \wedge \Box\top)) \in$ EK4 - 2,6 and the substitution $[q/\varphi]$.
8. $\varphi \wedge \Box\top \leftrightarrow q \wedge \Box\top \in \mathcal{T}h(M)$.
9. $\Box(\varphi \wedge \Box\top) \leftrightarrow \Box(q \wedge \Box\top) \in \mathcal{T}h(M)$ - 8 and equivalence.
10. $\Box\top \to (\Box q \leftrightarrow \Box\varphi) \in \mathcal{T}h(M)$ - 6,7,9 and PC.

End of proof of claim.

Hence by $x \in \|\Box\top\|$ and $x \in \|\Box q\|^M$ we see that $x \in \|\Box\varphi\|^M$, so $\|\varphi\|^M \in N(x)$. $\Box$

While things work smoothly in this case it is unfortunate that we do not see a precedent here for obtaining a 'generated' subframe. In general, the truth of a boxed formula really depends on global properties of the whole frame. For instance, it may happen that $(X, N, v) \vDash_y \Box\Box p$ and $(X, N, v) \nvDash_y \Box\Box q$ and the only reason this happens is because $(X, N, v) \nvDash_z \Box p \leftrightarrow \Box q$ holds only at this particular z. Hence, this particular z, which could be anywhere, becomes important to the notion of truth at the point y.

[2]The Propositional Calculus.

A.3 Horn Logics

We now move on to prove some purely syntactic features of a class of logics that includes EK4. Our principal tool is the notion of a metavaluation. The idea behind metavaluations is that we think of $\Box$-ed formulae as being basic building blocks and so we try to assign them truth values. We then hope that this is somehow consistent with the underlying logic.

Definition A.9. Let $\mathbb{E} = P \cup \{\Box\varphi \mid \varphi \in \mathcal{S}(P)\}$. A *metavaluation* is a map $v : \mathbb{E} \longrightarrow \{\top, \bot\}$. Each metavaluation extends uniquely to a map $v : \mathcal{S}(P) \longrightarrow \{\top, \bot\}$ using the usual requirements that $v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$ etc., except that we do not need to give the truth condition for $\Box\varphi$.

Definition A.10. We say that a metavaluation v *verifies* L iff $v(\varphi) = \top$ for all $\varphi \in L$.

We note that any metavaluation will verify all tautological formulae, so verifying that a metavaluation verifies a logic is quite straightforward:

Proposition A.11. *Let L be a classical logic defined by an axiom set Σ and sole rule of inference $\varphi \leftrightarrow \psi / \Box\varphi \leftrightarrow \Box\psi$. Then a metavaluation v verifies L iff*

1. $(\forall\varphi \in \Sigma)$ $(\forall$*substitutions* $\sigma)$ $[v(\varphi\sigma) = \top]$, *and*
2. *if* $\varphi \leftrightarrow \psi \in L$ *then* $v(\Box\varphi) = v(\Box\psi)$.

For this reason we will assume that all axiomatisations given in this chapter have $\varphi \leftrightarrow \psi/\Box\varphi \leftrightarrow \Box\psi$ as their only rule of inference (apart from modus ponens).

Before we properly embark on our study of Horn logics we make the following observation.

Lemma A.12. *Suppose that $\varphi \notin L$, then there is a metavaluation v which verifies L and $v(\varphi) = \bot$.*

Proof. Since $\varphi \notin L$, there is an $\mathcal{S}$-algebra $\underline{\underline{A}}$ and a valuation v' on $\underline{\underline{A}}$ such that $v'(\psi) = \top_A$ for all $\psi \in L$ and $v'(\varphi) \neq \top_A$. Let $u \subseteq A$ be an ultrafilter in $\underline{\underline{A}}$ which includes $\neg v'(\varphi)$, then define the metavaluation v by

$$v(q) = \begin{cases} \top & \text{if } v(q) \in u, \\ \bot & \text{otherwise.} \end{cases}$$

This can readily be shown to be a metavaluation which verifies L because $\psi \in L \Longrightarrow v'(\psi) = \top_A \Longrightarrow v(\psi) = \top$. Also, $v'(\varphi) \notin u$ so $v(\varphi) = \bot$. $\Box$

Now we can close in on our syntactic class of logics:

Definition A.13. We say that a formula is a *Horn clause* iff it is of the form

$$\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{n-1} \to \Box\psi,\ n \geqslant 1.$$

Using the name 'Horn' is a slight abuse of the established terminology, for a Horn clause usually allows arbitrary members of $\mathbb{E}$ however we are demanding that all components be boxed formulae.

Definition A.14. We say that a logic L is a *Horn logic* iff it can be axiomatised by a set of Horn clauses (in addition to the collection of tautologies).

Horn logics turn out to have two defining characteristics, those of being additive and unintensional. We will establish this relationship here.

Definition A.15. Let $\Box\mathcal{S}(P) = \{\Box\varphi \mid \varphi \in \mathcal{S}(P)\}$ $[\Diamond\mathcal{S}(P) = \{\Diamond\varphi \mid \varphi \in \mathcal{S}(P)\}]$ the set of *boxed* [*lozenged*] formulae respectively.

Definition A.16. We say that a logic is *unintensional* iff we can never have

$$\Box\varphi_0 \vee \cdots \vee \Box\varphi_{n-1} \in L \text{ or } \Diamond\varphi_0 \vee \cdots \vee \Diamond\varphi_{n-1} \in L.$$

Definition A.17. We say that a logic L is *additive* iff for all sets $\{v_\alpha \mid \alpha < \kappa\}$ of metavaluations which verify L, the metavaluation $v\downarrow$ defined by

$$v\downarrow(q) = \min\{v_\alpha(q) \mid i < \kappa\}$$

verifies L.

We can now verify the first half of the equivalence between Horn and unintensional additive logics.

Lemma A.18. *Suppose that L is a Horn logic. Then L is an unintensional and additive logic.*

Proof. First let Σ be a set of Horn clauses which axiomatise L. We show that the two properties hold:
<u>L is additive</u>

Let $\{v_\alpha \mid \alpha < \kappa\}$ be a collection of metavaluations which verify L, and set $v = v\downarrow$. We must show that v verifies L and this means that $v(\chi\sigma) = \top$ for all $\chi \in \Sigma$ and all substitutions σ and that $\varphi \leftrightarrow \psi \in L \Longrightarrow v(\Box\varphi) = v(\Box\psi)$.

<u>$(\forall\chi \in \Sigma)$ ($\forall$substitutions σ) $[v(\chi\sigma) = \top]$.</u>

Let $\chi \in \Sigma$ and let σ be a substitution. Thus

$$\chi\sigma = \Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{n-1} \rightarrow \Box\psi$$

for some $\varphi_0, \ldots, \varphi_{n-1}, \psi \in \mathcal{S}(P)$. Thus $v_\alpha(\chi\sigma) = \top$ for all $\alpha < \kappa$. Now suppose that $v(\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{n-1}) = \top$, i.e., $v(\Box\varphi_0) = \cdots = v(\Box\varphi_{n-1}) = \top$. Then $v_\alpha(\Box\varphi_0) = \cdots = v_\alpha(\Box\varphi_{n-1}) = \top$ for all $\alpha < \kappa$, and so $v_\alpha(\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{n-1}) = \top$ for all $\alpha < \kappa$, telling us that $v(\Box\psi) = \top$. We conclude that $v(\chi\sigma) = \top$.

$\underline{\varphi \leftrightarrow \psi \in L \Longrightarrow v(\Box\varphi) = v(\Box\psi).}$

Suppose that $\varphi \leftrightarrow \psi \in L$, so $\Box\varphi \leftrightarrow \Box\psi \in L$. Thus

$$(\forall\alpha < \kappa)\,[v_\alpha(\Box\varphi) = v_\alpha(\Box\psi)]\,.$$

We then have the following sequence of equivalences:

$$\begin{aligned} v(\Box\varphi) = \top &\Longleftrightarrow (\forall\alpha < \kappa)\,[v_\alpha(\Box\varphi) = \top] \\ &\Longleftrightarrow (\forall\alpha < \kappa)\,[v_\alpha(\Box\psi) = \top] \\ &\Longleftrightarrow v(\Box\psi) = \top \end{aligned}$$

<u>L is unintensional.</u>

Assume not. Thus we have 2 cases:

Case $\Box\varphi_0 \vee \cdots \vee \Box\varphi_{n-1} \in L$.

Define $v : \mathbb{E} \longrightarrow \{\top, \bot\}$ by $v(q) = \bot$ for $q \in \mathbb{E}$. We show that v verifies L.

<u>$(\forall\chi \in \Sigma)$ $(\forall$substitutions $\sigma)$ $[v(\chi\sigma) = \top]$.</u>

Let $\chi \in \Sigma$ and let σ be a substitution. Thus

$$\chi\sigma = \Box\varphi_0' \wedge \cdots \wedge \Box\varphi_{n-1}' \rightarrow \Box\psi$$

for some $\varphi_0', \ldots, \varphi_{n-1}', \psi \in \mathcal{S}(P)$. Thus $v(\Box\varphi_i') = \bot$ for $i < n$, and so $v(\Box\varphi_0' \wedge \cdots \wedge \Box\varphi_{n-1}') = \bot$ which tells us that

$$v(\Box\varphi_0' \wedge \cdots \wedge \Box\varphi_{n-1}' \rightarrow \Box\psi) = \top.$$

We can then conclude that $v(\chi\sigma) = \top$.

$\underline{\varphi \leftrightarrow \psi \in L \Longrightarrow v(\Box\varphi) = v(\Box\psi).}$

$v(\Box\varphi) = \bot = v(\Box\psi)$.

Thus

$$v(\Box\varphi_0 \vee \cdots \vee \Box\varphi_{n-1}) = \bot \vee \cdots \vee \bot = \bot$$

so $\Box\varphi_0 \vee \cdots \vee \Box\varphi_{n-1} \notin L$, a contradiction.

Case $\Diamond\varphi_0 \vee \cdots \vee \Diamond\varphi_{n-1} \in L$.

Define $v : \mathbb{E} \longrightarrow \{\top, \bot\}$ by $v(q) = \top$ for all $q \in \mathbb{E}$. We will show that v verifies L.

$\underline{(\forall\chi \in \Sigma)\ (\forall\text{substitutions } \sigma)\ [v(\chi\sigma) = \top].}$

Let $\chi \in \Sigma$ and let σ be a substitution. Thus

$$\chi\sigma = \Box\varphi'_0 \wedge \cdots \wedge \Box\varphi'_{n-1} \to \Box\psi$$

for some $\varphi'_0, \ldots, \varphi'_{n-1}, \psi \in \mathcal{S}(P)$. Thus $v(\Box\psi) = \top$ which tells us that

$$v(\Box\varphi'_0 \wedge \cdots \wedge \Box\varphi'_{n-1} \to \Box\psi) = \top.$$

We can then conclude that $v(\chi\sigma) = \top$.

$\underline{\varphi \leftrightarrow \psi \in L \Longrightarrow v(\Box\varphi) = v(\Box\psi).}$

$v(\Box\varphi) = \top = v(\Box\psi)$.

Thus

$$v(\Diamond\varphi_0 \vee \cdots \vee \Diamond\varphi_{n-1}) = \neg\top \vee \cdots \vee \neg\top = \bot$$

so $\Diamond\varphi_0 \vee \cdots \vee \Diamond\varphi_{n-1} \notin L$, a contradiction.

$\Box$

Proving the converse of this lemma is a little more tricky and we must detour through some technical lemmas.

Lemma A.19. *Suppose that the following formula is in L, an unintensional, additive logic:*

$$\chi = \bigvee_{k<r} p_k \vee \bigvee_{l<s} \neg q_l \vee \bigvee_{i<m} \neg\Box\varphi_i \vee \bigvee_{j<n} \Box\psi_j$$

where the $p_k \in P$, $k < r$ and $q_j \in P$, $j < n$ are all distinct. Then $m \neq 0$ and there exists a $j < n$ such that

$$\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{m-1} \to \Box\psi_j \in L.$$

Proof. We know, by rewriting $\chi \in L$, that

$$\bigwedge_{i<m} \Box\varphi_i \to \bigvee_{k<r} p_k \vee \bigvee_{l<s} \neg q_l \vee \bigvee_{j<n} \Box\psi_j \in L$$

There are then 6 cases and we will show that the following 5 cannot hold.

Case $n = 0$, $r = 0$, and $s = 0$.

Thus $m \neq 0$ and $\bigwedge_{i<m} \Box\varphi_i \to \bot \in L$. Thus $\bigvee_{i<m} \Diamond\neg\varphi_i \in L$ contradicting the unintensionality of L.

Case $\bigwedge_{i<m} \Box\varphi_i \to \Box\psi_j \in L$ for some $j < n$.

Then we are done.

Case $\bigwedge_{i<m} \Box\varphi_i \to p_k \in L$ for some $k < r$.

Then by making the substitution $\tau = [p_k/\bot]$ we get that $\bigwedge_{i<m} \Box\varphi_i\tau \to \bot \in L$ and we return to the argument of the first case.

Case $\bigwedge_{i<m} \Box\varphi_i \to \neg q_l \in L$ for some $l < s$.

Make the substitution $\tau = [q_l/\top]$ and proceed as in the first case.

Case $\bigwedge_{i<m} \Box\varphi_i \to \rho \notin L$ for all

$$\rho \in \Gamma := \{\Box\psi_j, p_k, \neg q_l \mid j < n, k < r, l < s\}.$$

Let $\sigma = [q_0/\neg q_0, \ldots, q_{s-1}/\neg q_{s-1}]$, and so $\bigwedge_{i<m} \Box\varphi_i \to \rho \notin L$ for all

$$\rho \in \Gamma' =: \{\Box\psi_j\sigma, p_k, q_l \mid j < n, k < r, l < s\} \subseteq \mathbb{E}.$$

(Otherwise, by applying σ again we would get back to our original formula.)

Thus, there is a metavaluation v_ρ, for all $\rho \in \Gamma'$, such that

$$v_\rho\left(\bigwedge_{i<m} \Box\varphi_i\sigma \to \rho\right) = \bot.$$

Let $v = \min\{v_\rho \mid \rho \in \Gamma'\}$. Thus $v\left(\bigwedge_{i<m} \Box\varphi_i\sigma\right) = \top$, as $v_\rho\left(\bigwedge_{i<m} \Box\varphi_i\sigma\right) = \top$, and $v(\rho) = \bot$ for all $\rho \in \Gamma'$, as $v_\rho(\rho) = \bot$ for all $\rho \in \Gamma'$.

Hence $v\left(\bigwedge_{i<m} \Box\varphi_i\sigma \to \bigvee_{\rho\in\Gamma'} \rho\right) = \bot$ and thus

$$\bigwedge_{i<m} \Box\varphi_i\sigma \to \bigvee_{\rho\in\Gamma'} \rho \notin L.$$

But, this is a substitution instance of a χ equivalent so $\chi \notin L$, a contradiction.

This case analysis has shown that the remaining case must hold. That is

$$\bigwedge_{i<m} \Box\varphi_i \to \Box\psi_j \in L$$

for some j, and we conclude that $m \neq 0$ since otherwise $\Box\psi_j \in L$, a contradiction to the unintensionality of L. $\Box$

Corollary A.20. *Let φ be a conjunction of elements of, or negated elements of, $\mathbb{E}$. Suppose that*

$$\varphi \to \Box\psi_0 \vee \cdots \vee \Box\psi_{n-1} \in L,$$

an unintensional additive logic. Then, for some $j < n$,

$$\varphi \to \Box\psi_j \in L.$$

Proof. Say that

$$\varphi = \bigwedge_{k<r} \neg p_k \wedge \bigwedge_{l<s} q_l \wedge \bigwedge_{i<m} \Box\varphi_i \wedge \bigwedge_{j'<u} \neg\Box\psi'_{j'}.$$

Thus $\varphi \to \Box\psi_0 \vee \cdots \vee \Box\psi_{n-1}$ is tautologically equivalent to

$$\bigvee_{k<r} p_k \vee \bigvee_{l<s} \neg q_l \vee \bigvee_{i<m} \neg\Box\varphi_i \vee \bigvee_{j'<u} \Box\psi'_{j'}. \vee \bigvee_{j<n} \Box\psi_j$$

which is in L. So by Lemma A.19 either

$$\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{m-1} \to \Box\psi'_{j'} \in L$$

for some $j' < u$, in which case $\neg\varphi \in L$ so $\varphi \to \Box\psi_0 \in L$ holds trivially, or

$$\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{m-1} \to \Box\psi_j \in L$$

for some $j < n$, and, since φ is stronger than the antecedent of this, we are done. $\Box$

The following lemma is a straightforward consequence of the definition of unintensionality

Lemma A.21. *Let L be an unintensional logic. Then L can have no theses of either of these forms:*

$$\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{n-1} \to \Diamond\psi_0 \vee \cdots \vee \Diamond\psi_{m-1}, \text{ and}$$
$$\Diamond\psi_0 \wedge \cdots \wedge \Diamond\psi_{m-1} \to \Box\varphi_0 \vee \cdots \vee \Box\varphi_{n-1}.$$

Lemma A.22. *Let L be an unintensional logic. Then $\Box\mathcal{S}(P)$ and $\Diamond\mathcal{S}(P)$ are both L-consistent.*

Proof. Assume not. Without loss of generality we take $\Diamond\mathcal{S}(P)$ to be inconsistent. Thus there exist $\varphi_0, \ldots, \varphi_{n-1}$ such that

$$\Diamond\varphi_0 \wedge \cdots \wedge \Diamond\varphi_{n-1} \to \bot \in L.$$

Hence $\Box\neg\varphi_0 \wedge \cdots \wedge \Box\neg\varphi_{n-1} \in L$ contradicting the unintensionality of L. $\Box$

We are now in a position to state and prove the converse of Lemma A.18

Lemma A.23. *Let L be an unintensional additive logic. Then L is a Horn logic.*

Proof. First note that L is unintensional, so $L \neq \mathcal{S}(P)$, and so L is consistent.

Now, for each $\chi \in L$ we can tautologically rewrite χ into disjunctive normal form, so

$$\chi \leftrightarrow \bigwedge_{i<k} \chi_i \in Taut,$$

where each χ_i is a disjunction of elements or negations of elements of $\mathbb{E}$. Set $DN(\chi) = \{\chi_i \mid i < k\}$ and then set $\Sigma_1 = \bigcup_{\chi \in L} DN(\chi)$, so Σ_1 axiomatises L.

We can assume that each $\chi \in \Sigma_1$ contains no redundancies so write

$$\chi = \bigvee_{k<r} p_k \vee \bigvee_{l<s} \neg q_s \vee \bigvee_{i<m} \neg\Box\varphi_i \vee \bigvee_{j<n} \Box\psi_j$$

where $p_k, q_l \in P$, and, since χ is non-tautological, no $p_k = q_l$.

We can immediately apply Lemma A.19 to this to get that $m \neq 0$ and that there is some $j < n$ such that $\chi' = \Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{m-1} \to \Box\psi_j \in L$. But $\chi' \to \chi \in Taut$, so

$$\{\chi' \mid \chi \in \Sigma_1\}$$

axiomatises L and it is a set of Horn clauses. $\Box$

Corollary A.24. *A logic L is Horn iff it is unintensional and additive.*

Horn logics actually have a very simple derivation structure as the following definition and lemma reveal.

Definition A.25. Let Σ be a set of Horn clauses, let L be a logic, and let $\Gamma \subseteq \Box\mathcal{S}(P)$. Then we define Γ_L^Σ, the set of Σ-consequences of Γ with respect to L, to be the closure of Γ in $\Box\mathcal{S}(P)$ under the rules:

$$\Box\varphi_0, \ldots, \Box\varphi_{n-1}/\Box\psi \text{ for } \Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{n-1} \to \Box\psi \in s\Sigma$$
$$\Box\varphi/\Box\psi \text{ if } \varphi \leftrightarrow \psi \in L,$$

where $s\Sigma = \{\chi\sigma \mid \chi \in \Sigma \text{ and } \sigma \text{ is a substitution}\}$.

Proposition A.26. *Suppose that L is a Horn logic. Then $\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{m-1} \rightarrow \Box\psi \in L$ iff*

$$\Box\psi \in \{\Box\varphi_0, \ldots, \Box\varphi_{m-1}\}^{\Sigma}_{L}$$

where Σ is a Horn clause axiomatisation of L.

Proof. Without loss of generality take Σ to be closed under substitutions.

($\Longleftarrow$) We prove this by induction on the derivation of $\Box\psi$ from $\{\Box\varphi_0, \ldots, \Box\varphi_{m-1}\}$ using the rules induced by Σ and L.

Base Cases: Suppose that $\Box\psi \in \{\Box\varphi_0, \ldots, \Box\varphi_{m-1}\}$. Thus $\Box\psi = \Box\varphi_i$ so $\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{m-1} \rightarrow \Box\psi \in L$ trivially holds.

Inductive Hypothesis: Suppose that $\Box\psi$ follows by application of a rule to $\Box\chi_0, \ldots, \Box\chi_{n-1}$ of which the results hold.

Inductive Step: We have two cases

Case The rule is $\Box\chi_0, \ldots, \Box\chi_{n-1}/\Box\psi \in \Sigma$

But $(\forall i < n)\,[\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{m-1} \rightarrow \Box\chi_i \in L]$, so $\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{m-1} \rightarrow \Box\psi \in L$.

Case The rule is $\Box\chi_0/\Box\psi$ where $\chi_0 \leftrightarrow \psi \in L$.

Thus $\Box\chi_0 \leftrightarrow \Box\psi \in L$ and $\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{m-1} \rightarrow \Box\chi_0 \in L$ by the inductive hypothesis, so $\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{m-1} \rightarrow \Box\psi \in L$.

($\Longrightarrow$) Suppose that $\Box\psi \notin \{\Box\varphi_0, \ldots, \Box\varphi_{m-1}\}^{\Sigma}_{L}$. Define a metavaluation $v : \mathbb{E} \longrightarrow \{\top, \bot\}$ by $v(q) = \top$ iff $q \in \{\Box\varphi_0, \ldots, \Box\varphi_{m-1}\}^{\Sigma}_{L}$.

Claim: v verifies L.

Proof of claim:

We need to check that $v(\varphi) = \top$ for each φ in the Horn clause axiomatisation Σ and that if $\varphi \leftrightarrow \psi \in L$ then $v(\Box\varphi) = v(\Box\psi)$. But each Horn clause in Σ is clearly verified and the equivalential condition is also immediate from the definition of Γ^{Σ}_{L}. *End of proof of claim.*

Thus $v(\Box\psi) = \bot$ and $v(\Box\varphi_i) = \top$ for all $i < m$, so $v(\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{m-1} \rightarrow \Box\psi) = \bot$, allowing us to conclude that $\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{m-1} \rightarrow \Box\psi \notin L$.

□

A.4 Amenable Logics

Now we move on to a more specialised class of logics which still contains EK4, and about which we can come to even more conclusions.

Definition A.27. We say that a Horn clause $\chi = \Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{n-1} \to \Box\psi$ is *amenable in a logic L* iff

$$\chi \in L \Longrightarrow (\exists i < n)\,[\psi \to \varphi_i \in L] \text{ or } (\exists\chi \in \mathcal{S}(P))\,[\psi \to \Box\chi]\,.$$

We say that a logic L is *amenable* iff all Horn clauses are amenable in L.

We say that a set of formulae Σ is *amenable* in L iff each $\chi \in \Sigma$ is amenable in L.

We say that a logic L has an *amenable axiomatisation* iff it is axiomatised by a set of Horn clauses and that set is amenable in L.

Proposition A.28. *A logic L is additive, unintensional and amenable iff it has an amenable axiomatisation.*

Proof. We prove the if and only if parts separately.

($\Longrightarrow$) Suppose that L is additive, unintensional and amenable. Thus L has a Horn axiomatisation by Corollary A.24. Since L is amenable each element of the Horn axiomatisations amenable and we are done.

($\Longleftarrow$) Suppose that L has an amenable axiomatisation. Call this axiomatisation Σ a set of Horn clauses which, without loss of generality, we assume to be closed under substitution. Thus by Corollary A.24 L is additive and unintensional. It remains to show that each $\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{n-1} \to \Box\psi \in L$ is amenable. So, suppose that this formula really is in L. Hence $\Box\psi \in \{\Box\varphi_0, \cdots, \Box\varphi_{n-1}\}^{\Sigma}_{L}$. We will prove by induction on the derivation of $\Box\psi$ that either

$$(\exists i < n)\,[\psi \to \varphi_i \in L] \text{ or } (\exists\chi \in \mathcal{S}(P))\,[\psi \to \Box\chi \in L]\,.$$

Base Cases: $\Box\psi \in \{\Box\varphi_0, \ldots, \Box\varphi_{n-1}\}$. thus $\psi = \varphi_i$ for some i and we are done.

Inductive Hypothesis: Assume that $\Box\psi$ results from an application of a rule to the formulae $\Box\psi_0, \ldots, \Box\psi_{m-1} \in \{\Box\varphi_0, \ldots, \Box\varphi_{n-1}\}^{\Sigma}_{L}$ and that the result holds for these.

Inductive Step: We have two cases.

Case The rule applied results from $\Box\psi_0 \wedge \cdots \wedge \Box\psi_{m-1} \to \Box\psi \in \Sigma$. Thus we have two subcases:

Subcase $(\exists i < m)\,[\psi \to \psi_i \in L]$.

The Inductive Hypothesis tells us that we have a further 2 subsubcases:

Subsubcase $(\exists j < n)\,[\psi_i \to \varphi_j \in L]$.
Hence $\psi \to \varphi_j \in L$ and we are done.

Subsubcase $(\exists \chi \in \mathcal{S}(P))\,[\psi_i \to \Box\chi \in L]$.
Hence $\psi \to \Box\chi \in L$ and we are done.

Subcase $(\exists \chi \in \mathcal{S}(P))\,[\psi \to \Box\chi \in L]$.
And we are done instantly.

Case The rule applied is $\Box\psi_0/\Box\psi$ where $\psi_0 \leftrightarrow \psi \in L$.
Here we know that either

$$(\exists i < n)\,[\psi_0 \to \varphi_i \in L] \text{ or } (\exists \chi \in \mathcal{S}(P))\,[\psi_0 \to \Box\chi \in L]$$

so by replacing provable equivalents in this we get that

$$(\exists i < n)\,[\psi \to \varphi_i \in L] \text{ or } (\exists \chi \in \mathcal{S}(P))\,[\psi \to \Box\chi \in L]$$

and we are done.

□

Thus we are able to tell almost instantly if a logic is additive, unintensional, and amenable as the first two conditions follow if we have a Horn axiomatisation and the amenability condition can frequently be obtained by inspection. Often we can detect amenability because it follows almost tautologically.

Definition A.29. A Horn clause $\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{n-1} \to \Box\psi$ is *patently amenable* iff

$$(\exists i < n)\,[\psi \to \varphi_i \in \mathsf{E}] \text{ or } (\exists \chi \in \mathcal{S}(P))\,[\psi \to \Box\chi \in \mathsf{E}]$$

A logic is additive, unintensional and amenable if it is axiomatised by a collection of patently amenable Horn clauses.

Example A.30. The following formulae are all patently amenable:

1. K: $\Box(p \to q) \to (\Box p \to \Box q)$, as $q \to (p \to q) \in Taut$.

2. 4: $\Box p \to \Box\Box p$, as $\Box p \to \Box p \in Taut$.

3. $\Box p \to \Box\Box^n p$ for $n > 1$, as $\Box^n p \to \Box\Box^{n-1} p \in Taut$.

4. $\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{n-1} \to \Box(\psi \wedge \Box\chi)$, as $\psi \wedge \Box\chi \to \Box\chi \in Taut$.

5. $\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{n-1} \to \Box(\varphi_0 \wedge \rho)$, as $\varphi_0 \wedge \rho \to \varphi_0 \in Taut$.

We are now able to draw two interesting conclusions about amenable, additive, unintensional (AAU) logics. The first of these speaks for itself:

Proposition A.31. *Suppose that L is AAU. Then*

$$\Box\psi_0 \wedge \cdots \wedge \psi_{n-1} \to \Box\top \in L \Longrightarrow (\exists i < n)\,[\varphi_i \in L]\,.$$

Proof. We know that either $(\exists i < n)\,[\top \to \varphi_i \in L]$ and we are done, or

$$(\exists \chi \in \mathcal{S}(P))\,[\top \to \Box\chi \in L]$$

and hence $\Box\chi \in L$, a contradiction to the unintensionality of L. □

Now we prove a result which may be relevant to the question of whether EK4 or similar logics have the finite model property.

The idea here is that we may start with a bunch of consistent formulae $\{\varphi_i \mid i < n\}$, say. Thus $(\forall i < n)\,[\neg\varphi_i \notin L]$. The set $\{\varphi_i \mid i < n\}$ may well be inconsistent, but if we are trying to build or force unusual models that have the potential to be counter examples to the finite model property, we may want a model which satisfies each φ_i in a possibly different place.

Define $\oplus_{i<n}\varphi_i = \neg\Box\Diamond\bot \wedge \bigwedge_{i<n}\Box\Diamond\varphi_i$.

Proposition A.32. *Suppose that Lis an AAU logic, and that each φ_i is consistent in L. Then $\oplus_{i<n}\varphi_i$ is L consistent.*

Proof. Assume not. Thus $\bigwedge_{i<n}\Box\Diamond\varphi_i \to \Box\Diamond\bot \in L$. Hence either $\Diamond\bot \to \Box\chi \in L$ for some χ (but this cannot happen) or there exists an $i < n$ such that $\Diamond\bot \to \Diamond\varphi_i \in L$. Thus $\Box\neg\varphi_i \to \Box\top \in L$ and hence by Lemma A.31 $\neg\varphi_i \in L$ a contradiction to each φ_i being L consistent. □

Proposition A.33. *Suppose that $M = (X, N, v) \vDash_x \oplus_{i<n}\varphi_i$. Then*

$$(\forall i < n)\,(\exists y_i \in X)\,[M \vDash_y \varphi_i]\,.$$

Proof. Let $i < n$. Thus $M \vDash_x \neg\Box(\Diamond\bot) \wedge \Box(\Diamond\varphi_i)$ and so $\|\Diamond\bot\|^M \notin N(x)$ and $\|\Diamond\varphi_i\|^M \in N(x)$. Hence $\|\Diamond\bot\|^M \neq \|\Diamond\varphi_i\|^M$ and so $\varnothing = \|\bot\|^M \neq \|\varphi_i\|^M$. Thus we can conclude that there is some $y_i \in X$ such that $y \in \|\varphi_i\|^M$, i.e., $M \vDash_y \varphi_i$. □

A.5 A Syntactic Result on EK4

We can say a few more things about EK4 which may be germane to the question of whether EK4 has the finite model property. We list these here.

For this section set $L =$ EK4 and recall that EK4 is an amenable, additive, unintensional logic.

Lemma A.34. *Suppose that $\varphi \leftrightarrow \Box^n\psi \in L$ and $\psi \leftrightarrow \Box^m\varphi \in L$ for some $n, m \in \omega$. Then[3] $\psi \leftrightarrow \varphi \in L$.*

Proof. Take $m, n > 0$ otherwise we are done. By n applications of the equivalence rule we get that $\Box^n\psi \leftrightarrow \Box^{m+n}\varphi \in L$. Thus $\varphi \leftrightarrow \Box^{m+n}\varphi \in L$ (1). Now, $\psi \to \Box^m\varphi \in L$ (2) by hypothesis and $\Box^m\varphi \to \Box^{m+n}\varphi \in L$ (3) by n applications of the 4 axiom. So $\psi \to \varphi \in L$ by (1), (2) and (3) together.

The implication $\varphi \to \psi \in L$ follows by symmetry. $\Box$

Lemma A.35. *Suppose that $\Box\varphi_0 \wedge \cdots \wedge \Box\varphi_{k-1} \to \Box\psi \in L$. Then if $\neg\varphi_i \to \Box^m\varphi_j \notin L$ for all $i, j < k$ and each $m \in \omega$, we have that there is some $n \in \omega$ and $i < k$ such that $\psi \leftrightarrow \Box^n\varphi_i \in L$.*

Proof. Suppose that $\neg\varphi_i \to \Box^m\varphi_j \notin L$ for all $i, j < n$, $m \in \omega$. We proceed by induction on the derivation of $\Box\psi$ in $\{\Box\varphi_0, \ldots, \Box\varphi_{k-1}\}^{\Sigma}_{L}$, where

$$\Sigma = \{\Box(\chi \to \rho) \wedge \Box\chi \to \Box\rho, \Box\chi \to \Box\Box\chi \mid \chi, \rho \in \mathcal{S}(P)\}.$$

Base Cases: If $\Box\psi \in \{\Box\varphi_0, \ldots, \Box\varphi_{k-1}\}$ then the result is immediate.
Inductive Hypothesis: Suppose that the result holds for

$$\Box\chi, \Box\eta \in \{\Box\varphi_0, \ldots, \Box\varphi_{k-1}\}^{\Sigma}_{L}.$$

That is, $\chi \leftrightarrow \Box^{n_1}\varphi_{i_1} \in L$ and $\eta \leftrightarrow \Box^{n_2}\varphi_{i_2} \in L$ for some $n_1, n_2 \in \omega$ and $i_1, i_2 < k$.
Inductive Step: Suppose that $\Box\psi$ follows from $\Box\chi$, $\Box\eta$ via one application of a rule derived from Σ. We have 3 cases:

Case $\eta = \chi \to \psi$, so $\Box\chi \wedge \Box\eta \to \Box\psi \in L$.

Thus $\neg\chi \to \eta \in L$ and so $\neg\Box^{n_1}\varphi_{i_1} \to \Box^{n_2}\varphi_{i_2} \in L$ by replacement by provable equivalents. Hence by L being unintensional either $n_1 = 0$ (in which case $\neg\varphi_{i_1} \to \Box^{n_2}\varphi_{i_2} \in L$) or $n_2 = 0$ (in which case $\neg\varphi_{1_2} \to \Box^{n_1}\varphi_{i_1} \in L$). Each possibility leads to a contradiction to our hypothesis, so this case cannot happen.

Case $\psi = \Box\chi$, so we are using $\Box\chi \to \Box\Box\chi \in L$.

But $\chi \leftrightarrow \Box^{n_1}\varphi_{i_1} \in L$ so $\Box\chi \leftrightarrow \Box^{n_1+1}\varphi_{i_1} \in L$ and we are done.

Case $\psi \leftrightarrow \chi \in L$ and the equivalence rule is applied.

Thus $\psi \leftrightarrow \Box^{n_1}\varphi_{i_1} \in L$.

$\Box$

[3] *This lemma actually holds in any extension of* E4.

Theorem A.36. *Suppose that $\Box\varphi \leftrightarrow \Box\psi \in L$. Then $\varphi \leftrightarrow \psi \in L$.*

Proof. We prove the following claim:
Claim: $\psi \leftrightarrow \Box^m\varphi \in L$ for some $m \in \omega$.
Proof of claim:

Assume not. Thus by Lemma A.35 and the fact that $\Box\varphi \rightarrow \Box\psi \in L$ we have that $\neg\varphi \rightarrow \Box^k\varphi \in L$ for some $k \in \omega$. But $\Box\psi \rightarrow \Box\varphi \in L$ so by L being an amenable logic we have two possibilities:

Case $\varphi \rightarrow \Box\chi \in L$ for some $\chi \in \mathcal{S}(P)$.

Thus $\neg\varphi \vee \varphi \rightarrow \Box^k\varphi \vee \Box\chi \in L$, so $\Box^k\varphi \vee \Box\chi \in L$ and so by the fact that L is unintensional $k = 0$. Thus $\neg\varphi \rightarrow \varphi \in L$, so $\varphi \in L$ telling us that $\Box\chi \in L$, a contradiction.

Case $\varphi \rightarrow \psi \in L$.

Use $\Box\varphi \rightarrow \Box\psi \in L$ and amenability to get the following two subcases.

Subcase $\psi \rightarrow \varphi \in L$.

And we are done.

Subcase $\psi \rightarrow \Box\chi \in L$ for some $\chi \in \mathcal{S}(P)$.

Since we have already narrowed down our consideration to the case where $\varphi \rightarrow \psi \in L$ we get that $\varphi \rightarrow \Box\chi \in L$. But this brings us back to our first case, which is contradictory.

End of proof of claim.

By symmetry we have also established that $\varphi \leftrightarrow \Box^n\psi \in L$ for some $n \in \omega$. Thus by Lemma A.34 we are done. □

Note: In the light of Lemma A.35 it might be thought that $\neg\varphi \rightarrow \Box^m\varphi \in L$ for $m > 0$ could never hold. This is not the case for consider $\varphi = \neg(\Box\top \wedge \Box\bot)$. Thus $\neg\varphi \rightarrow \Box\top \wedge \Box\bot \in L$. But it is almost immediate that $\Box\top \wedge \Box\bot \rightarrow \Box\varphi \in L$ ($\top \leftrightarrow (\bot \rightarrow \varphi) \in Taut$), so $\neg\varphi \rightarrow \Box\varphi \in L$.

A.6 EK4 and Ultrafilter Semantics

In this section we will consider one possible approach to proving the canonicity of EK4. Again, set $L =$ EK4.

The problem we encounter when trying to take our approach of Chapter 5 and apply it to EK4, is as follows: We may choose $Z \subseteq X^L$ to be represented by some c in the larger algebra, and so determine $J(c)$ by finding out what $K(c)$ represents. Unfortunately, our procedure already determined that $J(Z)$

was represented by some $c' \in C$ and there was no guarantee that c' was $K(c)$ or some other member of the range of K. So we *could not* use the fact that $K(c) \subseteq K(K(c))$ to conclude that $J(Z) = c' \subseteq K(c')$.

In the terminology[4] of ultrafilter semantics given in Chapter 7 (we will use the results and definitions of that chapter without reference) it is even clearer: It may be that $\|d(Z)\|_{\mathcal{U}} = Z$ and so by definition $J(Z) = \|\Box d(Z)\|_{\mathcal{U}}$ yet in all probability $\mu(d(J(Z))) \neq \mu(\Box d(Z))$ almost everywhere and hence we cannot argue that

$$\begin{aligned} J(Z) = \|\Box d(Z)\|_{\mathcal{U}} &\subseteq \|\Box\Box d(Z)\|_{\mathcal{U}} \\ &= \|\Box d(J(Z))\|_{\mathcal{U}} \\ &= J(J(Z)). \end{aligned}$$

However if we could guarantee that each set Z, which can be represented as some $\|\Box\varphi\|_{\mathcal{V}}$ for some $\varphi \in \mathcal{S}(Q)$ and ultrafilter $\mathcal{V}$, has the property that $\mu(d(Z)) \in \Box\mathcal{S}(P)\ \mathcal{U}$ a.e. with respect to our fixed ultrafilter, then we would have, by $J(Z) = \|\Box d(Z)\|_{\mathcal{U}}$, that $d(J(Z)) \in \Box\mathcal{S}(P)\ \mathcal{U}$ a.e. Thus

$$\begin{aligned} J(Z) &= \|d(J(Z)\|_{\mathcal{U}} \\ &\subseteq \|\Box d(J(Z))\|_{\mathcal{U}} \\ &= J(J(Z)), \end{aligned}$$

where this follows simply because

$$d(J(Z)) \in \Box\mathcal{S}(P) \Longrightarrow d(J(Z)) \to \Box d(J(Z)) \in L.$$

So, sets which can be represented as $\|\Box\varphi\|_{\mathcal{V}}$ are of special interest.

Definition A.37. We say that $Z \subseteq X^L$ is a *possible interior* iff there is a $\varphi \in \mathcal{S}(Q)$ and an ultrafilter $\mathcal{V}$ on Ω such that $\|\Box\varphi\|_{\mathcal{V}} = Z$.

This whole approach would fail if we had the possibility that an effable set was a possible interior yet was not an actual interior. That is $\|\Box\varphi\|_{\mathcal{U}} = \|\psi\|$ for ψ not equivalent to an element of $\Box\mathcal{S}(P)$. We do not want to have this because our ultrafilter semantics approach to canonicity demands that $\|\Box d(\|\psi\|)\|_{\mathcal{U}} = \|\Box\psi\|$ and so we do not want to have to deal with the complication of making $d(\|\psi\|)$ anything other than ψ or one of its L-equivalents.

The next two results show that we have no concerns on this score.

[4] We will use the substitutions μ of that chapter in their "functional form," that is, we will write '$\mu(\varphi)$' rather than '$\varphi(\ulcorner\mu\urcorner)$'. We also drop the modifier $\ulcorner\ \urcorner$.

Lemma A.38. *Let $\mathcal{U}$ be an ultrafilter on Ω, $\varphi \in \mathcal{S}(Q)$, and $\psi \in \mathcal{S}(P)$. Then*

$$\|\psi\| \subseteq \|\Box\varphi\|_{\mathcal{U}} \Longrightarrow \psi \to \mu(\Box\varphi) \in L\, \mathcal{U} \text{ a.e.}$$

Proof. Suppose that $\psi \to \mu(\Box\varphi) \notin L\,\mathcal{U}$ a.e. Say that ψ is in conjunctive normal form and that $\psi = \bigvee_{i<n} \psi_i$ where each ψ_i is a conjunction of members of $\mathbb{E}$. Thus

$$(\exists i < n)\, [\psi_i \to \mu(\Box\varphi) \notin L]\, \mathcal{U} \text{ a.e.}$$

and so by $\mathcal{U}$ an ultrafilter and n finite there is some i such that $\psi_i \to \mu(\Box\varphi) \notin L$ $\mathcal{U}$ a.e.

Now set

$$U = \{\mu \in \Omega \mid \psi_i \to \mu(\Box\varphi) \notin L\} \in \mathcal{U} \text{ and}$$
$$\Sigma = \{\psi_i, \neg\mu(\Box\varphi) \mid \mu \in U\}.$$

Claim: Σ is L-consistent.
Proof of claim:

Assume not. Thus there are $\mu_0, \ldots, \mu_{k-1} \in U$ such that

$$\psi_i \wedge \neg\mu_0(\Box\varphi) \wedge \cdots \wedge \neg\mu_{k-1}(\Box\varphi) \to \bot \in L.$$

Hence $\psi_i \to \Box\mu_0(\varphi) \vee \cdots \vee \Box\mu_{k-1}(\varphi) \in L$. Since ψ_i is a conjunction of members of $\mathbb{E}$ we can apply Lemma A.20 to get that there is some $j < k$ such that

$$\psi_i \to \Box\mu_j(\varphi) \in L.$$

But this is a contradiction to $\mu_j \in U$ (by the definition of U). *End of proof of claim.*

So let x be a maximal L-consistent set extending Σ. We then have that

$$x \in \|\psi_i\| \subseteq \|\psi\| \text{ and } (\forall \mu \in U)\, [\mu(\Box\varphi) \notin x]$$

and so $x \notin \|\Box\varphi\|_{\mathcal{U}}$. Thus x witnesses the fact that $\|\psi\| \not\subseteq \|\Box\varphi\|_{\mathcal{U}}$. ☐

Theorem A.39. *Let $\mathcal{U}$ be an ultrafilter on Ω, $\varphi \in \mathcal{S}(Q)$, and $\psi \in \mathcal{S}(P)$. Then*

$$\|\psi\| = \|\Box\varphi\|_{\mathcal{U}} \Longrightarrow \psi \leftrightarrow \mu(\Box\varphi) \in L\, \mathcal{U} \text{ a.e.}$$

Proof. Suppose that $\|\psi\| = \|\Box\varphi\|_{\mathcal{U}}$ and assume that

$$\psi \leftrightarrow \mu(\Box\varphi) \notin L\, \mathcal{U} \text{ a.e.} \tag{A.1}$$

By Lemma A.38 we then note that

$$\psi \to \Box\mu\,(\varphi) \in L\ \mathcal{U}\text{ a.e.} \tag{A.2}$$

One of the following must hold:

$$(\exists\chi \in \mathcal{S}\,(P))\,[\mu\,(\varphi) \to \Box\chi \in L]\ \mathcal{U}\text{ a.e.} \tag{A.3}$$
$$\text{or } (\forall\chi \in \mathcal{S}\,(P))\,[\mu\,(\varphi) \to \Box\chi \notin L]\ \mathcal{U}\text{ a.e.} \tag{A.4}$$

So take $U_0 \in \mathcal{U}$ to witness that

(A.1) and (A.2) and ((A.3) or (A.4)) holds.

<u>$(\forall U \in \mathcal{U})\,[\ \{\Box\mu\,(\varphi)\,, \neg\psi \mid \mu \in U\}$ is L-inconsistent $]$.</u>

Let $U \in \mathcal{U}$ and assume that $\{\Box\mu\,(\varphi)\,, \neg\psi \mid \mu \in U\}$ is L-consistent. So take x to be a maximal L-consistent extension of this set. Hence $\neg\psi \in x$ and $\mu\,(\Box\varphi) = \Box\mu\,(\varphi) \in x\ \mathcal{U}$ a.e. allowing us to conclude that $x \in \|\psi\|$, $x \notin \|\Box\varphi\|_{\mathcal{U}}$ a contradiction.

Claim: There are $\gamma_0, \ldots, \gamma_{n-1} \in \mathcal{S}\,(P)$ such that $\psi \leftrightarrow \Box\gamma_0 \wedge \cdots \wedge \Box\gamma_{n-1} \in L$.
Proof of claim:

By the L-inconsistency of $\{\Box\mu\,(\varphi)\,, \neg\psi \mid \mu \in U_0\}$ we get that there are $\mu_0, \ldots, \mu_{n-1} \in U_0$ such that $\Box\mu_0(\varphi) \wedge \cdots \wedge \Box\mu_{n-1}(\varphi) \wedge \neg\psi \to \bot \in L$, so if we put $\gamma_i = \mu_i(\varphi)$ for $i < n$, we get that $\Box\gamma_0 \wedge \cdots \wedge \Box\gamma_{n-1} \to \psi \in L$.

However $\psi \to \mu\,(\Box\varphi) \in L$ for all $\mu \in U_0$ so we have that $\psi \to \Box\gamma_i \in L$ for all $i < n$, giving us the other direction in our biconditional. *End of proof of claim.*
Thus we can conclude that, for all $\mu \in U_0$,

$$\Box\gamma_0 \wedge \cdots \wedge \Box\gamma_{n-1} \to \Box\mu(\varphi) \in L.$$

We then have two cases to consider

Case $(\exists i < n)\,[\gamma_i \leftrightarrow \mu\,(\varphi) \in L]\ \ \mathcal{U}$ a.e.

Thus $(\exists i < n)\,[\gamma_i \leftrightarrow \mu\,(\varphi) \in L\ \ \mathcal{U}$ a.e.$]$, so let $U \in \mathcal{U}$, $U \subseteq U_0$ be such that

$$(\forall\mu \in U)\,[\gamma_i \leftrightarrow \mu\,(\varphi) \in L]$$

and so by the equivalence rule

$$(\forall\mu \in U)\,[\Box\gamma_i \leftrightarrow \Box\mu\,(\varphi) \in L]\,.$$

But $\{\neg\psi, \Box\mu\,(\varphi) \mid \mu \in U\}$ is inconsistent so there are $\mu_0, \ldots, \mu_{m-1} \in U$

such that

$$\Box\mu_0(\varphi) \wedge \cdots \wedge \Box\mu_{m-1}(\varphi) \to \psi \in L.$$

Again, $\psi \to \Box\mu_0(\varphi) \wedge \cdots \wedge \Box\mu_{m-1}(\varphi) \in L$ holds by $U \subseteq U_0$ and using $\Box\gamma_i \leftrightarrow \Box\mu_j(\varphi) \in L$ we then get that $\psi \leftrightarrow \Box\mu_0(\varphi) \in L$. (Since the $\mu_j(\varphi)$ are pairwise equivalent, we can collapse the conjunction down to a single conjunct.)

But this contradicts our assumption that $\psi \leftrightarrow \Box\mu(\varphi) \notin L$ holds over all of U_0.

Case $(\forall i < n)\,[\gamma_i \leftrightarrow \mu(\varphi)] \notin L$ $\mathcal{U}$ a.e.

In this case suppose that this condition holds over $U_1 \in \mathcal{U}$, $U_1 \subseteq U_0$. Thus by the same argument as in the previous case we see that there are $\mu_0, \ldots, \mu_{m-1} \in U_1$ such that

$$\Box\mu_0(\varphi) \wedge \cdots \wedge \Box\mu_{m-1}(\varphi) \leftrightarrow \Box\gamma_0 \wedge \cdots \wedge \Box\gamma_{n-1} \in L.$$

Claim: We have the following two conditions:

1. For each $i < n$ there is a $\tau(i) < m$ and a $k(i) \in \omega$ such that $\gamma_i \leftrightarrow \Box^{k(i)}\mu_{\tau(i)}(\varphi) \in L$, and
2. For each $j < m$ there is a $\nu(j) < n$ and a $k'(j) \in \omega$ such that $\mu_j(\varphi) \leftrightarrow \Box^{k'(j)}\gamma_{\nu(j)} \in L$.

Proof of claim:

We proceed by considering 2 subcases—recall conditions (3) and (4) on our choice of U_0.

Subcase $(\forall\mu \in U_0)\,(\forall\chi \in \mathcal{S}(P))\,[\mu(\varphi) \to \Box\chi \notin L]$.

Using the fact that L is amenable, and our condition on $\mu(\varphi) \to \Box\chi$ we get instantly that for each $j < m$ there is a $\tau(j) < n$ such that

$$\mu_j(\varphi) \to \gamma_{\tau(j)} \in L$$

and for each $i < n$ there is a $\nu(i) < m$ such that $\gamma_i \to \mu_{\nu(j)}(\varphi) \in L$. Now put $k(i) = 0$, $k'(j) = 0$ uniformly, and we are done.

Subcase $(\forall\mu \in U_0)\,(\exists\chi \in \mathcal{S}(P))\,[\mu(\varphi) \to \Box\chi \in L]$.

Thus for all $j < m$ there is a $\chi_j \in \mathcal{S}(P)$ such that $\mu_j(\varphi) \to \Box\chi_j \in L$. Hence $\neg\mu_j(\varphi) \to \Box^k\mu_l(\varphi) \notin L$ for each $j, l < m$ and $k \in \omega$. (Otherwise: if $k > 0$ we would have $\Box^k\mu_j(\varphi) \vee \Box\chi_j \in L$, and we can eradicate the case of $k = 0$ by noting then that $\neg\mu_j(\varphi) \to \Box\chi_l \in$

L and so $\Box\chi_l \vee \Box\chi_j \in L$. Both these possibilities contradict the unintensionality of L.)

Since each $\gamma_i = \mu(\varphi)$ for some $\mu \in U_0$ we get by the same argument that $\neg\gamma_i \to \Box^k\gamma_l \notin L$ for each $i, l < n$, $k < \omega$. So by applying Lemma A.35 we have our desired result.

End of proof of claim.

We now note that $\nu \circ \tau : n \longrightarrow n$ so it must cycle somewhere. That is, there is some $i < n$ and an $r \in \omega$ such that $(\nu \circ \tau)^r(i) = i$. Set $j = \tau(i)$. We now know that

$$\gamma_i \leftrightarrow \Box^{k(i)}\mu_j(\varphi) \in L$$

so to apply Lemma A.34 we must show that

$$\mu_j(\varphi) \leftrightarrow \Box^{k''}\gamma_i \in L.$$

To this end, we have the following claim:

Claim: $(\forall r' \in \omega - \{0\})\,(\exists k'' \in \omega)\,\Big[\mu_j(\varphi) \leftrightarrow \Box^{k''}\gamma_{(\nu\circ\tau)^{r'}(i)} \in L\Big]$.

Proof of claim:

By induction on r'. The base case follows if we set $k'' = k'(j)$, so suppose that the result holds for r':

$$\mu_j(\varphi) \leftrightarrow \Box^{k''}\gamma_{(\nu\circ\tau)^{r'}(i)} \in L \tag{A.5}$$

We will now show that it holds for $r'+1$. First set $i' = (\nu \circ \tau)^{r'}(i)$. By the properties of the functions ν and τ we have that

$$\gamma_{i'} \leftrightarrow \Box^{k(i')}\mu_{\tau(i')}(\varphi) \in L \tag{A.6}$$

$$\mu_{\tau(i')}(\varphi) \leftrightarrow \Box^{k'(\tau(i'))}\gamma_{(\nu\circ\tau)(i')} \in L \tag{A.7}$$

and applying the equivalence rule k'' times to (A.6) and $k'' + k(i)$ times to (A.7) gives

$$\Box^{k''}\gamma_{i'} \leftrightarrow \Box^{k''+k(i')}\mu_{\tau(i')}(\varphi) \in L \tag{A.8}$$

$$\Box^{k''+k(i')}\mu_{\tau(i')}(\varphi) \leftrightarrow \Box^{k''+k(i')+k'(\tau(i'))}\gamma_{(\nu\circ\tau)(i')} \in L. \tag{A.9}$$

By applying the propositional calculus to (A.5), (A.8), and (A.9) and by noting that $(\nu \circ \tau)(i') = (\nu \circ \tau)^{r'+1}(i)$ we get

$$\mu_j(\varphi) \leftrightarrow \Box^{k'''}\gamma_{(\nu\circ\tau)^{r'+1}(i)} \in L$$

where $k''' = k'' + k(i') + k'(\tau(i'))$. *End of proof of claim.*

Set $r' = r$ in the above claim and we see that $\mu_j(\varphi) \leftrightarrow \Box^{k''}\gamma_i \in$ since $(\nu \circ \tau)^r(i) = i$. We are then able to apply Lemma A.34 to get that

$$\mu_j(\varphi) \leftrightarrow \gamma_i \in L$$

a contradiction to the case we are in.

$\Box$

Now we will show that the set of possible interiors can be realised as possible interiors with respect to one uniform ultrafilter $\mathcal{U}$.

Proposition A.40. *There is an ultrafilter $\mathcal{U}$ such that for all possible interiors Z,*

$$\|d(Z)\|_{\mathcal{U}} = Z \text{ and } d(Z) \in \Box\mathcal{S}(P)\ \mathcal{U}\text{ a.e.}$$

Proof. For each Z a possible interior let $\varphi_Z \in \mathcal{S}(Q)$, and $\mathcal{V}_Z$ be such that $Z = \|\Box\varphi_Z\|_{\mathcal{V}_Z}$. For each finite set F of possible interiors, and $v : F \longrightarrow \bigcup_{Z\in F}\mathcal{V}_Z$ satisfying $v(Z) \in \mathcal{V}_Z$, call the pair (F, v) *acceptable* and let

$$U(F,v) = \{\mu \in \Omega \mid (\forall Z \in F)\,(\exists \mu' \in v(Z))\,[\mu\,(d(Z)) = \mu'(\Box\varphi_Z)]\}\,.$$

It is then relatively easy to show that for each acceptable (F, v), $U(F, v) \neq \varnothing$ and that for each acceptable (F_1, v_1) and (F_2, v_2),

$$U(F_1, v_1) \cap U(F_2, v_2) \supseteq U(F', v'),$$

where $F' = F_1 \cup F_2$ and v is defined by

$$v'(Z) = \begin{cases} v_1(Z) & \text{if } Z \in F_1 - F_2 \\ v_2(Z) & \text{if } Z \in F_2 - F_1 \\ v_1(Z) \cap v_2(Z) & \text{if } Z \in F_1 \cap F_2. \end{cases}$$

So take $\mathcal{U}$ to be an ultrafilter which extends

$$\{U(F,v) \mid (F,V) \text{ is an acceptable pair}\}\,.$$

$\underline{Z = \|d(Z)\|_{\mathcal{U}}.}$

Let $z \in X$ and let Z be a possible interior. Then

$$\begin{aligned} z \in Z &\Longleftrightarrow z \in \|\Box\varphi_Z\|_{\mathcal{V}_Z} \\ &\Longleftrightarrow \{\mu' \mid \mu'(\Box\varphi_Z) \in z\} \in \mathcal{V}_Z \\ &\Longleftrightarrow \{\mu \mid \mu\,(d(Z)) \in z\} \in \mathcal{U} \end{aligned}$$

$$\iff z \in \|d(Z)\|_{\mathcal{U}}.$$

Where the third line can be obtained as follows:

($\Longrightarrow$) Set $F = \{Z\}$, $v(Z) = \{\mu' \mid \mu'(\Box\varphi_Z) \in z\} \in \mathcal{V}_Z$ and so

$$(\forall \mu \in U(F, v))\,(\exists \mu' \in v(Z))\,[\mu\,(d(Z)) = \mu'(\Box\varphi_Z) \in z]\,.$$

Thus $(\forall \mu \in U(F, v))\,[\mu\,(d(Z)) \in z]$.

($\Longleftarrow$) Suppose that $\{\mu' \mid \mu'(\Box\varphi_Z) \in z\} \notin \mathcal{V}_Z$. So set

$$F = \{Z\}\,,\; v(Z) = \{\mu' \mid \mu'(\Box\varphi_Z) \notin z\} \in \mathcal{V}_Z.$$

This then gives us that

$$(\forall \mu \in U(F, v))\,(\exists \mu' \in v(Z))\,[\mu\,(d(Z)) = \mu'(\Box\varphi_Z) \notin z]\,.$$

Thus $(\forall \mu \in U(F, v))\,[\mu\,(d(Z)) \notin z]$.

<u>$(\forall Z$ a possible interior$)\,[\mu\,(d(Z)) \in \Box\mathcal{S}\,(P)\ \mathcal{U}$ a.e.$]$.</u>

Let Z be a possible interior and let $\mu \in U(\{Z\}, \{\langle Z, \Omega\rangle\})$. Thus $\mu\,(d(Z)) = \mu'(\Box\varphi_Z)$ for some $\mu' \in \Omega$. Hence

$$\mu\,(d(Z)) = \Box\mu'(\varphi_Z) \in \Box\mathcal{S}\,(P)\,.$$

□

Unfortunately this proposition highlights a gaping hole in our progression so far: How can we ensure that our modifications still respect K? That is, how can we ensure that

$$(\forall Z, Y \subseteq X)\,[\|\Box d(\neg Z \cup Y)\|_{\mathcal{U}} \cap \|\Box d(Z)\|_{\mathcal{U}} \subseteq \|\Box d(Y)\|_{\mathcal{U}}]\,.$$

This author is currently unsure how to resolve this problem in this style of argument and so leaves the work at this point. His hope is that an astute reader can see how to bring these arguments to the desired conclusion.

Appendix B

An Error in Grove's Proof

With

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Nearly a decade has passed since GROVE [36] gave a semantics for the AGM postulates. The semantics, called sphere semantics, provided a new perspective of the area of study, and has been widely used in the context of theory or belief change. However, the soundness proof that GROVE gives in his paper is fallacious as it stands. In this note, we will point out the error and give two ways of repairing it.

We follow GROVE in matters of notation. To make this appendix self-contained, we start by rehearsing the notation we need. Let L be a propositional language. Let M_L be the set of all maximally consistent sets of sentences of L.[1] If A is any sentence, $|A|$ is the set of all members of M_L containing A. If $X \subseteq M_L$, $t(X)$ is the theory $\bigcap X$. If Σ is a set of sentences, $\mathrm{Cn}(\Sigma)$ is the set of logical consequences of Σ. If T is a theory in L, $c(A)$ is a certain subset of M_L—intuitively, the smallest "sphere" containing all extensions of T, some of which contain A—and $T + A$ is defined as $t(|A| \cap c(A))$. For future reference, T/A, the *expansion* of T by A, is $\mathrm{Cn}(T \cap \{A\})$.

We can now state the mistake in GROVE's proof. On p. 161, in verifying the postulate +7 two successive lines of the proof are:

$$\begin{aligned} |A| \cap |B| \cap c(A) &\subseteq |A| \cap |B| \cap c(A \cap B) \\ |B| \cap |T + A| &\subseteq |\mathrm{Cn}(\{A, B\})| \cap c(A \cap B) \end{aligned}$$

The left-hand side of this step relies on the fact that:

$$|T + A| \subseteq |A| \cap c(A)$$

[1] Alternatively, M_L can be taken as set of all models of the language, but we follow GROVE here.

i.e.:

$$|t(|A| \cap c(A))| \subseteq |A| \cap c(A).$$

But in general $|t(X)| \not\subseteq X$. Let X be the class of all maximal consistent sets minus any one of them. $t(X)$ is the set of all tautologies, for any non-tautology is not in at least two maximal consistent sets. Hence $|t(X)| = M_L$. A similar problem besets the verification of +8.

A simple solution to this problem is to require that every sphere in M_L be elementary, i.e., of the form $|B|$ for some sentence B. In this case, all the subsets of M_L appearing in the argument are elementary, and for any such set, X, $|t(X)| = X$,[2] so the argument goes through. This is the approach taken in TANAKA's [96]. The constraint means, in effect, that every sphere represents a theory that is finitely axiomatisable. Depending on how, exactly, one conceptualises spheres, this may or may not be a reasonable constraint.

Alternatively, GROVE's proof may be repaired through the following sequence of observations.

Lemma B.1. *Suppose that $(T + A)/B$ is consistent. Then $c(A) = c(A \wedge B)$.*

Proof. The left to right inclusion follows by the minimality of the sets picked out by c. To show the reverse inclusion it is enough to show that $c(A) \cap |A \wedge B| \neq \varnothing$. Assume not. Thus $|{\sim}(A \wedge B)| \supseteq c(A) \supseteq c(A) \cap |A|$, which gives us that ${\sim}(A \wedge B) \in t(c(A)) \subseteq t(c(A) \cap |A|) = T + A$. But $A \in T + A$ by postulate +2, so ${\sim}B \in T + A$ contradicting the consistency of $(T + A)/B$. □

We now observe how an expansion of a theory $t(X)$ is related to the underlying set X.

Lemma B.2. *For $X \subseteq M_L$, $t(X)/B = t(X \cap |B|)$.*

Proof. For the ($\subseteq$) part, let $C \in t(X)/B$ which means that $B \to C \in t(X)$, i.e., $(\forall x \in X)\,[B \to C \in x]$. Then it is immediate that $(\forall x \in X \cap |B|)\,[C \in x]$ giving $C \in t(X \cap |B|)$.

For the reverse inclusion, let $C \notin t(X)/B$ which tells us that $B \to C \notin t(X)$ and so $(\exists x \in X)\,[B \to C \notin x]$. By maximality of x we get that $x \in |B|$ and $C \notin x$, so $x \in X \cap |B|$ and $C \notin x$. Thus $C \notin t(X \cap |B|)$. □

We can now get at the desired result through the following theorem which is proved by a simple derivation.

Theorem B.3. *If ${\sim}B \notin T + A$ then $(T + A)/B = T + A \wedge B$.*

---

[2] See, e.g. BELL and SLOMSON's [4, p. 141].

Proof. Suppose that $\sim B \notin T + A$. Then $(T + A)/B$ is consistent. So by Lemma 1, $c(A \wedge B) = c(A)$.

$$\begin{aligned}(T + A)/B &= t(c(A) \cap |A|)/B \\ &= t(c(A) \cap |A| \cap |B|) \\ &= t(c(A) \cap |A \wedge B|) \\ &= t(c(A \wedge B) \cap |A \wedge B|) \\ &= T + A \wedge B.\end{aligned}$$

□

Postulates +7 and +8 are now simple corollaries of the result already in hand.

Corollary B.4. *+7: $T + A \wedge B \subseteq (T + A)/B$.*

Proof. If $\sim B \notin T + A$ then the theorem gives us full equality and if $\sim B \in T + A$ then $(T + A)/B$ is inconsistent so the result holds trivially. □

Corollary B.5. *+8: If $\sim B \notin T + A$ then $(T + A)/B \subseteq T + A \wedge B$.*

Proof. Trivial. □

Appendix C

Revising Some Basic Proofs in Belief Revision

ABSTRACT: This appendix reobtains GROVE's results [36] through means which are hopefully clearer and more illustrative of the underlying notions. Also, the proofs are worked through in enough detail that possible errors in GROVE's proof are clearly avoided. The paper concludes with a discussion on elementary (or closed) systems of spheres and it notes that each revision function can be given by such a system.

C.1 Introduction

In 1988 ADAM GROVE [36] published his paper which gives the sphere semantics for simple belief revision. His approach was to start with the conditions mandated by the earlier paper of ALCHOURRÓN, GÄRDENFORS and MAKINSON (AGM) [1] and then to show that these are equivalent to his system of spheres modeling. Once he had that he went on to show that the notion of an epistemic entrenchment relation also gives rise to a notion of revision which is equivalent to that given by the AGM postulates.

To prove that the epistemic relation notion is equivalent to a belief revision operation GROVE showed that it was naturally equivalent to a system of spheres. Unfortunately the proofs he gave were long and involved and hence difficult to follow. Moreover, the central proof of the 'completeness' of the sphere semantics with respect to $+$, the notion of revision, is obscure to the point that it is impossible to follow—for this author anyway. In particular in [36, p. 162] he writes:

> Let S′ be the class of all nonempty subsets U of M_L satisfying
>
> 1. $\forall u \in U, \exists A \in F\colon u \in |T + A|$
> 2. If $|A| \cap U \neq \varnothing$ for any $A \in F$ then $|T + A| \subseteq U$.

> Let S, the system of spheres, be $S' \cup \{M_L\}$ ...

Then he makes the statement:

> It is quite straightforward to show that S is indeed a system of spheres centered on $|T|$.

This is quite a strong statement because it means that the linearity and minimality properties can be derived quickly. Linearity by itself is complex and the only way it could possibly be shown is by appealing to the properties of $+$, and in effect, the author suspects, we would have to prove that underlying it all is an entrenchment relation on F.

Certainly the author sees no way that we can *straightforwardly* show that S is a system of spheres.

The idea that there has to be an entrenchment relation underlying GROVE's proof suggests that maybe it would be easier to first prove that entrenchment is equivalent to the AGM notion of $+$, then use that to produce systems of spheres. Thankfully, this turns out to be the case and we get a development which is, in the author's opinion, clean and elegant and the purpose of this paper is to present this development here. We will see the proof in full detail, taking in the result of an earlier paper by GRAHAM PRIEST, KOJI TANAKA and the author [65] which corrects an error in GROVE's proof of the soundness of his sphere semantics.

We will finish off the paper by making a few comments on *Closed Systems of Spheres* where each sphere is a closed set in the Stone space, or, more revealingly, is a set defined by a set of formulae—an 'elementary' set.

C.2 Notation and Basic Definitions

We will follow GROVE [36] almost entirely here. Let F be the set of all formulae over some (countable) language and, at least, the usual binary connectives of $\wedge, \vee, \rightarrow, \sim, \leftrightarrow, \top$, and $\perp$. We use A, B, C, D to represent elements of F. Let L be a logic over F, (i.e., $L \subseteq F$) which we take to be classical (all propositional tautologies are in L, and it is closed under detachment (from $A \rightarrow B \in L$ and $A \in L$ infer that $B \in L$)) and compact (having no infinitary rules of inference).[1] We choose $K \subseteq F$ to be the set of consistent formulae—namely $\{A \mid \sim A \notin L\}$.

We then take $T \subseteq F$ to be a fixed consistent theory; that is: T is closed under detachment and $L \subseteq T$. Set $\mathbb{T} = \{T' \subseteq F \mid T' \text{ is a theory}\}$. If T' is a

[1] For the results of this paper it would be enough for L to just be a theory (over some classical compact logic) which all other theories must extend.

theory and $A \in F$ then T'/A is the result of adding A to T' and closing under logical consequence, i.e., $T'/A = \mathrm{Cn}\,(T' \cap \{A\})$.[2]

Definition C.1. A revision function[3] $+$ for a theory T is a function $+ : F \longrightarrow \mathbb{T}$ which satisfies (where we write $T + A$ for $+(A)$):

(+2) $A \in T + A$,

(+3) $\sim A \notin T \Longrightarrow T + A = T/A$,

(+4) $A \in K \Longrightarrow T + A$ is consistent,

(+5) $A \leftrightarrow B \in L \Longrightarrow T + A = T + B$,

(+7) $T + A \wedge B \subseteq (T + A)/B$, and

(+8) $\sim B \notin T + A \Longrightarrow (T + A)/B \subseteq T + A \wedge B$.

Note that if $\sim B \in T + A$ then $(T + A)/B = F$ so we can rewrite (+7) and (+8) as

(+7,+8) $\sim B \notin T + A \Longrightarrow (T + A)/B = T + A \wedge B$.

In this paper we will also be looking at the canonical Stone space of L, namely the set

$$M_L = \{x \mid x \subseteq F \text{ is a maximal } L\text{-consistent set}\}\,.$$

Of course, this space has some topological structure but this will not become important until Section C.6.

As GROVE points out, [36, p. 158], each theory gives rise to a subset of M_L and each subset of M_L gives rise to a theory. Formally:

Definition C.2. We define the functions $|\cdot| : \mathbb{T} \cup F \longrightarrow \mathcal{P}(M_L)$ and $t : \mathcal{P}(M_L) \longrightarrow \mathbb{T}$ as follows:

1. For $T' \in \mathbb{T}$ and $A \in F$ set

$$|T'| = \{x \in M_L \mid T' \subseteq x\}\,, \text{ and}$$
$$|A| = \{x \in M_L \mid A \in x\}\,.$$

---

[2] $\mathcal{P}$ is the powerset function and $\mathrm{Cn} : \mathcal{P}(F) \longrightarrow \mathcal{P}(F)$ is the function which closes sets under L-consequence.

[3] Whereas GROVE [36, p. 175] takes the revision function $+$ to be a function of all theories T, we take the no more restricted view that each theory T can have its own revision function.

2. For $S \subseteq M_L$ set

$$t(S) = \{A \in F \mid (\forall x \in S)\,[A \in x]\} = \bigcap S.$$

GROVE gives the following useful, yet straightforward results [36, p. 158]:

1. $(\forall T' \in \mathbb{T})\,[t\,(|T'|) = T']$,[4]
2. $(\forall S \subseteq M_L)\,[S \neq \varnothing \Longrightarrow t\,(S)$ is consistent],[5]
3. $(\forall A \in F)\,(\forall S \subseteq M_L)\,[t\,(S \cap |A|) = t\,(S)\,/A]$,[6]
4. $(\forall S, S' \subseteq M_L)\,[S \subseteq S' \Longrightarrow t\,(S') \subseteq t\,(S)]$, and
5. $(\forall T', T'' \in \mathbb{T})\,[T' \subseteq T'' \Longrightarrow |T''| \subseteq |T'|]$.

One final point of notation, we take ω to be the first infinite ordinal and for the purposes of this appendix we are using it only as the set of all natural numbers.

C.3 Entrenchment Relations and A-consistency

C.3.1 Entrenchment Relations

We are now ready to look at the first of the 'equivalents' of the revision function and, unlike GROVE [36], we start with the entrenchment relation.

Definition C.3. Let $\leqslant$ be a relation on F. We say that $\leqslant$ is an entrenchment relation starting at T iff[7]

($\leqslant$1) $\leqslant$ is connected:

$$(\forall A, B \in F)\,[A \leqslant B \text{ or } B \leqslant A]\,,$$

[4] GROVE points out that the logic must be compact for this to hold, a fact which we already assume.

[5] Compare this with the proof of (+4) given in Theorem C.40.

[6] This result is actually restated and proven in full in Lemma C.39.

[7] GROVE points out that ($\leqslant$1) is redundant, being derivable from ($\leqslant$2) and ($\leqslant$3): To see this, note that $A \to (A \vee B) \vee B \in L$ so either $B \leqslant A$ and we are done, or $A \vee B \leqslant A$. So take $A \vee B \leqslant A$. Also, $B \to (A \vee B) \vee A \in L$ so either $A \leqslant B$ and we are done or $A \vee B \leqslant B$. So take $A \vee B \leqslant B$. Now, $A \vee B \to A \vee B \in L$ so either $A \leqslant A \vee B$ and we are done (together with $A \vee B \leqslant B$ and ($\leqslant$2) this gives us $A \leqslant B$) or $B \leqslant A \vee B$ and we are done again (together with $A \vee B \leqslant A$ and ($\leqslant$2) this gives $B \leqslant A$).

($\leqslant$2) $\leqslant$ is transitive:

$$(\forall A, B \in F)\,[A \leqslant B \text{ and } B \leqslant C \Longrightarrow A \leqslant C],$$

($\leqslant$3) For all $A, B, C \in F$:

$$A \to B \vee C \in L \Longrightarrow (B \leqslant A \text{ or } C \leqslant A),$$

($\leqslant$4) The formulae $A \in F$ such that $\sim A \notin T$ are precisely the $\leqslant$-minimal formulae:

$$(\forall A \in F)\,[(\forall B \in F)\,[B \leqslant A] \Longleftrightarrow \sim A \notin T],$$

($\leqslant$5) The elements of $F - K$[8] are precisely the $\leqslant$-maximal formulae:

$$(\forall A \in F)\,[(\forall B \in F)\,[A \leqslant B] \Longleftrightarrow A \notin K].$$

We write $A < B$ iff $B \nleqslant A$ (since then $A \leqslant B$).

With this definition in hand we will prove a few simple properties of these types of relations. Take $\leqslant$ to be an arbitrary entrenchment relation starting at T.

Proposition C.4. *For all $A, B, C \in F$:*

$$A \vee B \leqslant C \Longrightarrow A \leqslant C \text{ or } B \leqslant C.$$

Proof. Let $A, B, C \in F$ and suppose that $A \vee B \leqslant C$. Since L is classical we have that $A \vee B \to A \vee B \in L$ so $A \leqslant A \vee B$ or $B \leqslant A \vee B$ by ($\leqslant$3) and so by ($\leqslant$2) either $A \leqslant C$ or $B \leqslant C$. □

Proposition C.5. $(\forall A \in F)\,[A \leqslant A]$.

Proof. Let $A \in F$. We have that $A \to A \vee A \in L$, so by ($\leqslant$3) either $A \leqslant A$ or $A \leqslant A$. □

Proposition C.6. $(\forall A, B \in F)\,[A \to B \in L \Longrightarrow B \leqslant A]$.

Proof. Let $A, B \in F$ and suppose that $A \to B \in L$. Hence $A \to B \vee B \in L$ so $B \leqslant A$ by ($\leqslant$3). □

Using transitivity we get an immediate corollary to this result:

Corollary C.7. *If $A \to B \in L$ and $B \leqslant C$ then $A \leqslant C$.*

[8] For X and Y two sets, we take $X - Y$ to be the difference of the two sets, namely $\{x \in X \mid x \notin Y\}$.

C.3.2 A-consistency

In the next section we will show that $\leqslant$ gives rise, in a natural way, to a revision function. The basic intuition is that we use a formula A, and its position in the entrenchment relation, to beef up our notion of consistency and, thus, our notion of consequence.

We start by making precise our new (or extended) notion of consistency.

Definition C.8. Let $\Sigma \subseteq F$ and let $A \in K$. We say that Σ is *A-consistent* iff

$$(\forall B \in \mathrm{Cn}(\Sigma))\,[B \leqslant A].$$

Note that this notion extends our standard notion of consistency.

Proposition C.9. *If $\Sigma \subseteq F$ is A-consistent for $A \in K$ then Σ is consistent.*

Proof. Suppose that Σ is inconsistent. Then $\bot \in \mathrm{Cn}(\Sigma)$ so $\bot \leqslant A$ so by ($\leqslant$5), $A \notin K$. □

Since elements of $F - K$ are inconsistent to begin with we make an exception for them:

Definition C.10. Let $\Sigma \subseteq F$ and $A \in F - K$. We say that Σ is *A-consistent* iff Σ is consistent.

We can now derive a few straightforward results about consistency, where we let Σ range over subsets of F.

Proposition C.11. *Suppose that $A \leqslant B$. If Σ is A-consistent then it is also B-consistent.*

Proof. Let Σ be A-consistent. We will show that it is B-consistent.

The cases where A or B are inconsistent are straightforward.

$\underline{(\forall C \in \mathrm{Cn}(\Sigma))\,[C \leqslant B].}$

Let $C \in \mathrm{Cn}(\Sigma)$. Thus $C \leqslant A$ by Σ A-consistent and so $C \leqslant A \leqslant B$ giving $C \leqslant B$.

□

Proposition C.12. *Suppose that $\Sigma_1 \subseteq F$ is A-consistent and suppose further that $\Sigma_2 \subseteq \Sigma_1$. Then Σ_2 is A-consistent.*

Proof. In the case were $A \in F - K$, A-consistency is just consistency and so the result is immediate. So take $A \in K$. We must show that every $B \in \mathrm{Cn}(\Sigma_2)$ satisfies $B \leqslant A$. But this is immediate from the A-consistency of Σ_1 and the fact that $\mathrm{Cn}(\Sigma_2) \subseteq \mathrm{Cn}(\Sigma_2)$. □

We say that $\Sigma \subseteq F$ is A-inconsistent iff it is not A-consistent.

Lemma C.13. *Suppose that $\Sigma \cup \{B_1, \ldots, B_n\}$ is A-inconsistent for $A \in K$, then there exists a $C \in \mathrm{Cn}(\Sigma)$ such that*

$$C \wedge B_1 \wedge \cdots \wedge B_n \nleqslant A.$$

Proof. We have that there exists a

$$D \in \mathrm{Cn}(\Sigma \cup \{B_1, \ldots, B_n\})$$

such that $D \nleqslant A$. The deduction theorem for classical logic then tells us that there exists a $C \in \mathrm{Cn}(\Sigma)$ such that

$$C \wedge B_1 \wedge \cdots \wedge B_n \rightarrow D \in L$$

so then by Proposition C.6 we have that

$$D \leqslant C \wedge B_1 \wedge \cdots \wedge B_n.$$

By ($\leqslant$3) we then conclude that

$$C \wedge B_1 \wedge \cdots \wedge B_n \nleqslant A.$$

□

C.3.3 *A*-consequence.

So with this notion of inconsistency we can define a notion of consequence by taking our cue from the indirect deduction theorem.

Definition C.14. For $\leqslant$ an entrenchment relation and $A \in F$ we define $\vdash_A$, the A-consequence, as follows:

$$\Sigma \vdash_A B \text{ iff } \Sigma \cup \{\sim B\} \text{ is } A\text{-inconsistent.}$$

Remark C.15. This definition highlights the need to treat elements of $F - K$ separately for if we took a set Σ to be $\perp$-consistent iff each $B \in \mathrm{Cn}(\Sigma)$ satisfied $B \leqslant \perp$ then every set would be $\perp$-consistent and so nothing would '$\perp$-follow' from any set.

This is a very natural concept corresponding to our usual conceptions of a $\vdash$ as we see below:

Proposition C.16. *For $\Sigma_1 \subseteq \Sigma_2 \subseteq F$, and $A, B \in F$,*

$$\Sigma_1 \vdash_A B \Longrightarrow \Sigma_2 \vdash_A B.$$

Proof. Suppose that $\Sigma_1 \vdash_A B$. Then $\Sigma_1 \cup \{\sim B\}$ is A-inconsistent and so $\Sigma_2 \cup \{\sim B\}$ is A-inconsistent by the contrapositive of Proposition C.12. Hence $\Sigma_2 \vdash_A B$. □

Proposition C.17. *For $A \in F$, $B \in \Sigma \subseteq F$, $\Sigma \vdash_A B$.*

Proof. In the case where $A \notin K$ the result is immediate. So take $A \in K$.

Assume not, i.e., $\Sigma \nvdash_A B$. Thus $\Sigma \cup \{\sim B\}$ is A-consistent and so $B \wedge \sim B \leqslant A$ which tells us that $\bot \leqslant A$ and we can conclude that $A \notin K$, a contradiction. □

Proposition C.18. *For $\Sigma \subseteq F$, and $A, B, C \in F$,*

$$\Sigma \vdash_A B \rightarrow C \Longleftrightarrow \Sigma \cup \{B\} \vdash_A C.$$

Proof. If $A \notin K$ then this corresponds to the usual notion of consequence and the result holds. So take $A \in K$.

($\Longrightarrow$) Suppose that $\Sigma \vdash_A B \rightarrow C$. Thus $\Sigma \cup \{\sim(B \rightarrow C)\}$ is A-inconsistent, so $\exists D \in \mathrm{Cn}(\Sigma)$ such that $D \wedge \sim(B \rightarrow C) \nleqslant A$ by Lemma C.13, i.e., $D \wedge B \wedge \sim C \nleqslant A$. Hence $\Sigma \cup \{B\} \cup \{\sim C\}$ is A-inconsistent. We can then conclude that $\Sigma \cup \{B\} \vdash_A C$.

($\Longleftarrow$) Suppose that $\Sigma \cup \{B\} \vdash_A C$. Thus $\Sigma \cup \{B\} \cup \{\sim C\}$ is A-inconsistent, so by Lemma C.13 $\exists D \in \mathrm{Cn}(\Sigma)$ such that $D \wedge B \wedge \sim C \nleqslant A$. Hence $D \wedge \sim(B \rightarrow C) \nleqslant A$, telling us that $\Sigma \cup \{\sim(B \rightarrow C)\}$ is A-inconsistent, and we conclude that $\Sigma \vdash_A B \rightarrow C$.

□

We include the following results to indicate that we really are dealing with the full notion of a $\vdash$.

Proposition C.19. *Suppose that $\Sigma \vdash_A B$. Then there exists a $\Delta \subseteq \Sigma$, which is finite and $\Delta \vdash_A B$.*

Proof. As usual, we need only look at the case where $A \in K$.

We have that $\Sigma \cup \{\sim B\}$ is A-inconsistent. Hence $\exists D \in \mathrm{Cn}(\Sigma)$ such that $D \wedge \sim B \nleqslant A$, but $D \in \mathrm{Cn}(\Delta)$ for some finite $\Delta \subseteq \Sigma$. Thus $\Delta \cup \{\sim B\}$ is A-inconsistent and so $\Delta \vdash_A B$. □

Proposition C.20. *If $\Sigma \vdash_B C$ and $A \leqslant B$ then $\Sigma \vdash_A C$.*

Proof. Suppose that $\Sigma \vdash_B C$ and $A \leqslant B$. Thus $\Sigma \cup \{\sim C\}$ is B-inconsistent so by Proposition C.11 $\Sigma \cup \{\sim C\}$ is A-inconsistent, allowing us to conclude that $\Sigma \vdash_A C$. □

Since $\perp$ is an $\leqslant$-maximal formula and since $\vdash_\perp$ is just the usual $\vdash_L$ we get the following corollary:

Corollary C.21. *If* $\Sigma \vdash_L B$ *then* $\Sigma \vdash_A B$.

Proposition C.22. *For* $\Sigma \subseteq F$ *and* $A, B, C \in F$,

$$\Sigma \vdash_A B \to C \text{ and } \Sigma \vdash_A B \Longrightarrow \Sigma \vdash_A C.$$

Proof. Suppose that $\Sigma \vdash_A B \to C$ and $\Sigma \vdash_A B$. Thus $\Sigma \cup \{B, \sim C\}$ is A-inconsistent and $\Sigma \cup \{\sim B\}$ is A-inconsistent. Hence there are $D_1, D_2 \in \mathrm{Cn}(\Sigma)$ such that

$$D_1 \wedge B \wedge \sim C \nleqslant A \quad \text{and} \quad D_2 \wedge \sim B \nleqslant A.$$

Set $D = D_1 \wedge D_2 \in \mathrm{Cn}(\Sigma)$ and note that

$$D \wedge \sim C \to (D_1 \wedge B \wedge \sim C) \vee (D_2 \wedge \sim B) \in L,$$

so by ($\leqslant$3), $D_1 \wedge B \wedge \sim C \leqslant D \wedge \sim C$ or $D_2 \wedge \sim B \leqslant D \wedge \sim C$, and then transitivity will tell us that $D \wedge \sim C \nleqslant A$. Hence $\Sigma \vdash_A C$. □

This shows then that:

Proposition C.23. *For* $\Sigma \subseteq F$ *and* $A \in F$,

$$\{B \in F \mid \Sigma \vdash_A B\}$$

is a theory.

Now we have our final result showing the $\vdash$ nature of $\vdash_A$.

Proposition C.24. *Suppose that* $\Sigma, \Delta \subseteq F$, $A \in F$ *and that* $(\forall B \in \Delta)\,[\Sigma \vdash_A B]$ *and* $\Delta \vdash_A C$, *then* $\Sigma \vdash_A C$.

Proof. Without loss of generality we can take Δ to be finite—Proposition C.19—i.e.,

$$\Delta = \{D_1, \ldots, D_n\} \vdash_A C,$$

and so by our version of the Deduction Theorem, Proposition C.18,

$$\varnothing \vdash_A D_1 \wedge \cdots \wedge D_n \to C.$$

so by Proposition C.16 we have that

$$\Sigma \vdash_A D_1 \wedge \cdots \wedge D_n \to C.$$

But, $D_1, \ldots, D_n \in \{B \in F \mid \Sigma \vdash_A B\}$ which is a theory, so $C \in \{B \in F \mid \Sigma \vdash_A B\}$ so $\Sigma \vdash_A C$. □

Now we will note a result which will be useful in our sphere construction of Subsection C.5.3. This result is little more than the Lindenbaum construction of basic logic.

Lemma C.25. *Suppose that $\Sigma \subseteq F$ is A-consistent and $B \in F$. Then either $\Sigma \cup \{B\}$ or $\Sigma \cup \{\sim B\}$ is A-consistent.*

Proof. Assume not, i.e., $\Sigma \vdash_A \sim B$ and $\Sigma \vdash_A B$. But $\{\sim B, B\} \vdash_L \perp$ so $\{\sim B, B\} \vdash_A \perp$ by Corollary C.21. Thus by Proposition C.24 $\Sigma \vdash_A \perp$, so Σ is A-inconsistent, a contradiction. □

Lemma C.26 (Lindenbaum). *Let $A \in F$. Every A-consistent set Σ can be extended to a maximal consistent set x which is also A-consistent.*

Proof. Let $\{B_i\}_{i \in \omega}$ be an enumeration of F. Define a sequence $\{\Sigma_i\}_{i \in \omega}$ of subsets of F as follows:

$$\Sigma_0 = \Sigma, \text{ and}$$
$$\Sigma_{n+1} = \Sigma_n \cup \begin{cases} \{B_n\} & \text{if } \Sigma_n \cup \{B_n\} \text{ is } A\text{-consistent,} \\ \{\sim B_n\} & \text{otherwise.} \end{cases}$$

Thus each Σ_n, for $n \in \omega$, is A-consistent (appealing to Lemma C.25) and so

$$x = \bigcup_{n \in \omega} \Sigma_n$$

is A-consistent (by Proposition C.19) and also maximal consistent since for each n, one of B_n or $\sim B_n$ is in x. □

C.4 Entrenchment Relations and Revision Functions

Now that we have the basic properties of entrenchment relations and their attendant notions of consistency in hand, we can move on to showing how entrenchment gives rise to revision functions and vice versa.

C.4.1 Soundness

Suppose that we are given an entrenchment relation $\leqslant$ on F. How then could we construct a revision function? More precisely, given some $A \in F$, how can we revise T by A to create $T +_{\leqslant} A$? A reasonable starting point is $\{A\}$ itself and then we must somehow decide what has to be in our theory. Our natural definition is then:

Definition C.27. $T +_{\leqslant} A = \{B \in F \mid \{A\} \vdash_A B\}$.

We immediately have that this is indeed a theory (Proposition C.23) and the other properties follow with little work.

Theorem C.28. *The function* $+_{\leqslant}$*, as defined above, satisfies (+2) to (+5) and (+7,+8).*

Proof. Let $A, B \in F$.
(+2) $A \in T +_{\leqslant} A$.

$\{A\} \vdash_A A$ by Proposition C.17 so the result follows.

(+3) $\sim A \notin T \Longrightarrow T +_{\leqslant} A = T/A$.

Suppose that $\sim A \notin T$. Then we have the following sequence of equivalences:

$$\begin{aligned} B \in T +_{\leqslant} A &\Longleftrightarrow B \in \{C \in F \mid \{A\} \vdash_A C\} \\ &\Longleftrightarrow \{A\} \vdash_A B \\ &\Longleftrightarrow \varnothing \vdash_A A \to B \\ &\Longleftrightarrow \sim(A \to B) \nleqslant A \\ &\Longleftrightarrow \sim(A \to B) \text{ is not } \leqslant\text{-minimal} \\ &\Longleftrightarrow \sim\sim(A \to B) \in T \\ &\Longleftrightarrow A \to B \in T \\ &\Longleftrightarrow B \in T/A \end{aligned}$$

(+4) $A \in K \Longrightarrow T +_{\leqslant} A$ is consistent.

Suppose that $T +_{\leqslant} A$ is inconsistent and assume that $A \in K$. Consider the following derivation:

$$\begin{aligned} \bot \in T +_{\leqslant} A &\Longrightarrow \{A\} \vdash_A \bot \\ &\Longrightarrow \varnothing \vdash_A A \to \bot \\ &\Longrightarrow \varnothing \vdash_A \sim A \end{aligned}$$

$$\Longrightarrow {\sim}{\sim}A \nleqslant A$$
$$\Longrightarrow A \nleqslant A$$

The first of these conditions holds by $T +_{\leqslant} A$ inconsistent and so the last condition holds but this contradicts Proposition C.5.

(+5) $A \leftrightarrow B \in L \Longrightarrow T +_{\leqslant} A = T +_{\leqslant} B$.

Suppose that $A \leftrightarrow B \in L$. Then the following sequence of equivalences establishes the equality of the two sets $T +_{\leqslant} A$ and $T +_{\leqslant} B$:

$$\begin{aligned} D \in T +_{\leqslant} A &\Longleftrightarrow \{A\} \vdash_A D \\ &\Longleftrightarrow \varnothing \vdash_A A \to D \\ &\Longleftrightarrow \varnothing \vdash_B A \to D \\ &\Longleftrightarrow \varnothing \vdash_B B \to D \\ &\Longleftrightarrow \{B\} \vdash_B D \\ &\Longleftrightarrow D \in T +_{\leqslant} B \end{aligned}$$

The third line follows by $A \leqslant B$, $B \leqslant A$ and Proposition C.20, and the fourth line by Proposition C.23.

(+7,+8) ${\sim}B \notin T +_{\leqslant} A \Longrightarrow (T +_{\leqslant} A)/B = T +_{\leqslant} A \wedge B$.

Suppose that ${\sim}B \notin T +_{\leqslant} A$, and so $\{A\} \nvdash_A {\sim}B$. Hence $\varnothing \nvdash_A A \to B$, so ${\sim}(A \to {\sim}B) \leqslant A$ which tells us that $A \wedge B \leqslant A$, but trivially $A \leqslant A \wedge B$ (by $A \wedge B \to A \in L$ and Proposition C.6). We then get the following sequence of equivalences which establishes the equality of the sets $(T +_{\leqslant} A)/B$ and $T +_{\leqslant} A \wedge B$.

$$\begin{aligned} D \in (T +_{\leqslant} A)/B &\Longleftrightarrow B \to D \in T +_{\leqslant} A \\ &\Longleftrightarrow \{A\} \vdash_A B \to D \\ &\Longleftrightarrow \varnothing \vdash_A A \to (B \to D) \\ &\Longleftrightarrow \varnothing \vdash_A A \wedge B \to D \\ &\Longleftrightarrow \varnothing \vdash_{A \wedge B} A \wedge B \to D \\ &\Longleftrightarrow \{A \wedge B\} \vdash_{A \wedge B} D \\ &\Longleftrightarrow D \in T +_{\leqslant} A \wedge B \end{aligned}$$

The fifth line follows by $A \leqslant A \wedge B \leqslant A$ and Proposition C.20.

□

We should mention that our notion of $+_{\leqslant}$ corresponds exactly to that given by GROVE [36, p. 164]:

Proposition C.29. *For $A \in F$,*

$$T +_{\leqslant} A = \{B \in F \mid (A \wedge B) < (A \wedge {\sim}B)\}.$$

Proof. We have the following series of equivalences:

$$\begin{aligned} B \in T +_{\leqslant} A &\iff \{A\} \vdash_A B \\ &\iff A \wedge {\sim}B \nleqslant A \\ &\iff A < A \wedge {\sim}B \\ &\iff A \wedge B < A \wedge {\sim}B \end{aligned}$$

This last line is derived as follows:

($\Longrightarrow$) Suppose that $A < A \wedge {\sim}B$. Consider the following derivation:

$$\begin{aligned} A \leqslant A &\Longrightarrow A \wedge (B \vee {\sim}B) \leqslant A \\ &\Longrightarrow (A \wedge B) \vee (A \wedge {\sim}B) \leqslant A \\ &\Longrightarrow A \wedge B \leqslant A \text{ or } A \wedge {\sim}B \leqslant A \\ &\Longrightarrow A \wedge B \leqslant A < A \wedge {\sim}B \end{aligned}$$

The third line follows by Proposition C.4 and the last line follows because we have supposed that $A < A \wedge {\sim}B$ and so the second disjunct cannot hold.

Since $A \leqslant A$ holds we must have $A \wedge B \leqslant A < A \wedge {\sim}B$.

($\Longleftarrow$) Suppose that $A \wedge B < A \wedge {\sim}B$. Now, $A \leqslant A \wedge B$ by $A \wedge B \to A \in L$ and Proposition C.6, thus $A < A \wedge {\sim}B$.

□

So, in a natural way, we have established the soundness of our modelling.

C.4.2 Completeness

Demonstrating completeness is a slightly more tricky task as we must show not only that a revision function gives rise to a natural entrenchment but also that this entrenchment corresponds to the revision function through the construction of Subsection C.4.1.

How will we define our $\leqslant$? Consider two formulae $A, B \in F$. The formula A will be more entrenched than B if we are more likely to believe it when given the opportunity, and the ideal opportunity is when we are forced to believe one or the other. Formally:

Definition C.30. Given a revision function $+$, we define $\leqslant_+$ on F as follows:

$$A \leqslant_+ B \Longleftrightarrow_{\mathrm{df}} \begin{cases} T + A \vee B \subseteq T + A & \text{if } A \in K, \\ B \notin K & \text{if } A \notin K. \end{cases}$$

Where unambiguous throughout this subsection, we will suppress the $+$ in $\leqslant_+$.

We had to take the special case of $A \notin K$ since otherwise $T + A$ would be inconsistent and so $T + A \vee B \subseteq T + A$ would trivially hold.

We must verify that $\leqslant$ really is a relation of the appropriate type.

Lemma C.31. *If $A, B \in F$ then either $A \leqslant B$ or $B \leqslant A$.*

Proof. If $A \notin K$ then $T + A \vee B = T + B$ and so $B \leqslant A$, and similarly for $B \notin K$.

So take $A, B \in K$.

Claim: A or B is consistent with $T + A \vee B$.

Proof of claim:

Assume not, i.e., $\sim B, \sim A \in T + A \vee B$. Thus $\sim A \wedge \sim B \in T + A \vee B$ and so $\sim(A \vee B) \in T + A \vee B$, giving $T + A \vee B$ inconsistent which contradicts $A, B \in K$ and (+4). *End of proof of claim.*

Without loss of generality take A to be consistent with $T + A \vee B$. Thus

$$(T + A \vee B)/A = T + (A \vee B) \wedge A = T + A,$$

by (+7,+8) and (+5). Hence $T + A \vee B \subseteq T + A$. □

Lemma C.32. *If $A \leqslant B$ and $B \leqslant C$ then $A \leqslant C$.*

Proof. The cases where any of $A, B, C \notin K$ are straightforward so suppose that $A \leqslant B$, $B \leqslant C$, and that $A, B, C \in K$. By Lemma C.31 one of $A \leqslant B \vee C$ or $B \vee C \leqslant A$.

Case $A \leqslant B \vee C$.

Thus $T + A \vee (B \vee C) \subseteq T + A$, and so

$$\begin{aligned} T + A \vee C &= (T + A \vee B \vee C)/A \vee C \\ &\subseteq (T + A)/A \vee C \\ &= T + A. \end{aligned}$$

So $A \leqslant C$.

Case $B \vee C \leqslant A$.

Thus

$$T + A \vee B \vee C \subseteq T + B \vee C \subseteq T + B,$$

with the last inclusion following by $B \leqslant C$. So B (and hence $A \vee B$) is consistent with $T + A \vee B \vee C$. Thus

$$\begin{aligned} T + A \vee B \vee C &\subseteq (T + A \vee B \vee C)/A \vee B \\ &= T + A \vee B \\ &\subseteq T + A, \end{aligned}$$

with the last line following because $A \leqslant B$. We can now conclude that $A \leqslant B \vee C$ and we are left back in our earlier case which showed that $A \leqslant C$.

□

Theorem C.33. *The relation $\leqslant_+$, as defined, is an entrenchment relation starting at T, i.e., it satisfies conditions ($\leqslant$1), ($\leqslant$2), ($\leqslant$3), ($\leqslant$4), and ($\leqslant$5).*

Proof. The conditions ($\leqslant$1) and ($\leqslant$2) are just Lemmas C.31 and C.32 respectively.

<u>($\leqslant$3) $A \to B \vee C \Longrightarrow B \leqslant A$ or $C \leqslant A$.</u>

Suppose that $A \to B \vee C \in L$.

Case $A \notin K$.

Thus by definition $B \leqslant A$ (and $C \leqslant A$).

Case $A \in K$ (so $B \vee C \in K$).

Assume that $B \nleqslant A$ and $C \nleqslant A$, so $A \leqslant B$ and $A \leqslant C$.

Claim: B is consistent with $T + A \vee B$ or C is consistent with $T + A \vee C$

Proof of claim:

Assume not, i.e.,

$$\begin{aligned} {\sim}B &\in T + A \vee B \subseteq T + A, \text{ and} \\ {\sim}C &\in T + A \vee C \subseteq T + A. \end{aligned}$$

Hence ${\sim}B \wedge {\sim}C \in T + A$ or more succinctly ${\sim}(B \vee C) \in T + A$. But $A \in T + A$ so $B \vee C \in T + A$ telling us that $T + A$ is inconsistent, a contradiction to (+4). *End of proof of claim.*

Without loss of generality take B to be consistent with $T + A \vee B$. Hence, by (+7,+8),

$$\begin{aligned} (T + A \vee B) &\subseteq (T + A \vee B)/B \\ &= T + (A \vee B) \wedge B \\ &= T + B. \end{aligned}$$

So, $B \leqslant A$ as required.

$(\leqslant 4)$ $(\forall A \in F)\,[(\forall B \in F)\,[A \leqslant B] \iff {\sim}A \notin T]$.

Let $A \in F$. If $A \notin K$ the result is immediate since $A \nleqslant \top$ and ${\sim}A \in T$.

($\Longrightarrow$) Suppose that ${\sim}A \in T$ and we want to show that ${\sim}A < A$. Assume not. Thus $A \leqslant {\sim}A$ and so $T + A \vee {\sim}A \subseteq T + A$. Hence

$$T = T/A \vee {\sim}A = T + A \vee {\sim}A =\subseteq T + A$$

and so ${\sim}A \in T \subseteq T + A$ and $A \in T + A$ by (+2) so $\bot \in T + A$. Hence $A \notin K$, a contradiction.

($\Longleftarrow$) Suppose that ${\sim}A \notin T$.

$(\forall B \in F)\,[A \leqslant B]$

Let $B \in F$. Since A is consistent with T, $A \vee B$ is consistent with T and so by (+7,+8)

$$\begin{aligned} T + A \vee B &= T/A \vee B \\ &\subseteq (T/A \vee B)/A \\ &= T/(A \vee B) \wedge A \\ &= T/A = T + A. \end{aligned}$$

That is: $A \leqslant B$.

$(\leqslant 5)$ $(\forall B \in F)\,[B \leqslant A] \iff {\sim}A \in L$.

Let $A \in F$.

($\Longrightarrow$) Suppose that $(\forall B \in F)\,[B \leqslant A]$. Thus $\bot \leqslant A$. Hence by Definition C.30 $A \notin K$, i.e., ${\sim}A \in L$.

($\Longleftarrow$) Suppose that ${\sim}A \in L$. Then $(\forall B \in F)\,[B \leqslant A]$ holds by Definition C.30.

□

Now that we have established that $\leqslant_+$ really is an entrenchment relation it remains to show that $+ = +_{\leqslant_+}$ which allows us to justifiably conclude that $+$ and $\leqslant_+$ are 'equivalent'.

Theorem C.34. *For all $A \in F$, $T + A = T +_{\leqslant_+} A$.*

Proof. Let $A, B \in F$. If $A \notin K$ then $T + A = F = T +_{\leqslant} A$ trivially so take $A \in K$. Then

$$\begin{aligned} B \in T +_{\leqslant} A &\iff \{A\} \vdash_A B \\ &\iff A \wedge \sim B \nleq A \\ &\iff T + (A \wedge \sim B) \vee A \nsubseteq T + A \wedge \sim B \\ &\iff \sim B \text{ is not consistent with } T + A \\ &\iff B \in T + A. \end{aligned}$$

The second to last equivalence is derived as follows:

($\Longrightarrow$) Suppose that $\sim B$ is consistent with $T + A$. Thus

$$(T + A)/\sim B = T + A \wedge \sim B, \text{so}$$
$$T + (A \wedge \sim B) \vee A = T + A \subseteq T + A \wedge \sim B.$$

($\Longleftarrow$) Suppose that $T + A \subseteq T + A \wedge \sim B$. Thus $A \wedge \sim B$, and hence $\sim B$, are consistent with $T + A$.

The above sequence of equivalences establishes the desired equality. □

C.5 Systems of Spheres

In this section we introduce the Systems of Spheres of GROVE [36] and show that they easily emerge from the entrenchment relations of the previous section.

C.5.1 Grove's Definitions

Grove [36, pp. 158–159] defines a system of spheres as follows:

Definition C.35. A collection S of subsets of M_L is a *system of spheres centered on* $X \subseteq M_L$ iff it satisfies:

(S1) S is totally ordered by $\subseteq$; that is

$$(\forall U, V \in \mathrm{S})\,[U \subseteq V \text{ or } V \subseteq U],$$

(S2) X is the $\subseteq$-minimum of S; that is

$$(\forall U \in \mathrm{S})\,[X \subseteq U],$$

(S3) $M_L \in \mathrm{S}$, and

(S4) For $A \in F$, if there is a sphere in S, which intersects $|A|$, then there is a smallest sphere in S intersecting $|A|$, or more precisely

$$(\forall A \in F)\,[(\exists U \in \mathrm{S})\,[U \cap |A| \neq \varnothing] \Longrightarrow (\exists V \in \mathrm{S})\,[V \cap |A| \neq \varnothing \text{ and } (\forall V' \in \mathrm{S})\,[V' \subsetneq V \Longrightarrow V' \cap |A| = \varnothing]]].$$

This last condition will sometimes be referred to as "The Minimality Condition."

Then GROVE goes on to define a revision function $+_{\mathrm{S}}$ for each each system of spheres S as follows:

Definition C.36. If S is a system of spheres and $A \in K$ then *The Minimal Intersection Function* for S evaluated at A is

$$c(A) = \min_{\subseteq} \{U \in \mathrm{S} \mid U \cap |A| \neq \varnothing\},$$

and if $A \in F - K$ then $c(A) = M_L$.

When we are dealing with more than one system of spheres, we subscript c by its appropriate system.

Since $M_L \in \mathrm{S}$ and because of condition (S4) we see that $c(A)$ is defined for all $A \in F$. Then we can go on to give the revision function induced in this way.

Definition C.37. If S is a system of spheres centered on $|T|$ then $+_{\mathrm{S}}$ is defined as follows:

$$T +_{\mathrm{S}} A = t\,(c(A) \cap |A|)\,.$$

As before we must show that this system is both 'sound' and 'complete' with respect to the class of valid revision functions.

C.5.2 Soundness

GROVE's original proof of this [36, Theorem 1] was found to be in error by GRAHAM PRIEST and KOJI TANAKA and this subsection reiterates the contents of their paper [65][9] where the problem and solution are presented. Throughout this section take $+$ to be $+_{\mathrm{S}}$ for some system of spheres S.

[9]Also our Appendix B.

We can now state the mistake in GROVE's proof. On p. 161 in [36], in verifying the postulate (+7) (and by similarity of proof (+8)), two successive lines of the proof are:

$$|A| \cap |B| \cap c(A) \subseteq |A| \cap |B| \cap c(A \wedge B)$$
$$|B| \cap |T + A| \subseteq |\mathrm{Cn}(\{A, B\})| \cap c(A \wedge B)$$

The left-hand side of this step relies on the fact that:

$$|T + A| \subseteq |A| \cap c(A)$$

i.e.:

$$|t(|A| \cap c(A))| \subseteq |A| \cap c(A).$$

But in general $|t(X)| \nsubseteq X$. Let X be the class of all maximal consistent sets minus any one of them. Then $t(X)$ is the set of all tautologies, for any non-tautology is not in at least two maximal consistent sets. Hence $|t(X)| = M_L$. A similar problem besets the verification of (+8).

A simple solution to this problem is to require that every sphere in M_L be elementary, i.e., of the form $|T'|$ for some theory T'. In this case, all the subsets of M_L appearing in the argument are elementary, and for any such set, X, $|t(X)| = X$, so the argument goes through. A similar approach is taken in [96] where the spheres are actually taken to be the points 'in' finitely axiomatisable theories—i.e., a sphere is of the form $|A|$ for some $A \in F$.[10]

Alternatively, GROVE's proof may be repaired through the following sequence of observations where we take A, B, C to range over formulae in F.

Lemma C.38. *Suppose that $(T + A)/B$ is consistent. Then $c(A) = c(A \wedge B)$.*

Proof. The left to right inclusion follows by minimality and to show the reverse inclusion it is enough to show that $c(A) \cap |A \wedge B| \neq \varnothing$. Assume not. Thus $|\sim(A \wedge B)| \supseteq c(A) \supseteq c(A) \cap |A|$, which gives us that $\sim(A \wedge B) \in t(c(A)) \subseteq t(c(A) \cap |A|) = T + A$. But $A \in T + A$ by postulate (+2), so $\sim B \in T + A$ contradicting the consistency of $(T + A)/B$. □

We now observe how an expansion of a theory $t(X)$ is related to the underlying set X. This is just a restatement of one of GROVE's original observations.

Lemma C.39. *For $X \subseteq M_L$, $t(X)/B = t(X \cap |B|)$.*

Proof. For the ($\subseteq$) part, let $C \in t(X)/B$ which means that $B \rightarrow C \in t(X)$, i.e. $(\forall x \in X)[B \rightarrow C \in x]$. Then it is immediate that $(\forall x \in X \cap |B|)[C \in x]$ giving $C \in t(X \cap |B|)$.

[10] See, e.g., [4, p. 141] for a similar use of the term 'elementary'.

For the reverse inclusion, let $C \notin t(X)/B$ which tells us that $B \to C \notin t(X)$ and so $(\exists x \in X)[B \to C \notin x]$. By maximal consistency of x we get that $x \in |B|$ and $C \notin x$, so $x \in X \cap |B|$ and $C \notin x$. Thus $C \notin t(X \cap |B|)$. □

We can now get at the desired result of soundness.

Theorem C.40. *The function $+_S$, as defined, is a revision function, i.e., $+_S$ satisfies conditions (+2), (+3), (+4), (+5) and (+7,+8).*

Proof. Let $A, B \in F$

(+2) $A \in T + A$.

Let $x \in c(A) \cap |A|$, thus $A \in x$. So $A \in T + A$.

(+3) $\sim A \notin T \Longrightarrow T + A = T/A$.

Suppose that $\sim A \notin T$. Thus $|A| \cap |T| \neq \varnothing$ and so $c(A) = |T|$. Thus

$$T + A = t(c(A) \cap |A|) = t(|T| \cap |A|) = T/A.$$

(+4) $A \in K \Longrightarrow T + A$ is consistent.

Suppose that $A \in K$. Thus $M_L \cap |A| \neq \varnothing$, so by minimality $c(A) \cap |A| \neq \varnothing$. So let $x \in c(A) \cap |A|$. Thus $T + A = t(c(A) \cap |A|) \subseteq t(\{x\}) = x$ so $T + A$ is consistent.

(+5) $A \leftrightarrow B \in L \Longrightarrow T + A = T + B$.

Suppose that $A \leftrightarrow B \in L$. Thus $c(A) = c(B)$ and $|A| = |B|$, telling us that

$$T + A = t(c(A) \cap |A|) = t(c(B) \cap |B|) = T + B.$$

(+7,+8) $\sim B \notin T + A \Longrightarrow (T + A)/B = T + A \wedge B$.

Suppose that $\sim B \notin T + A$, so Lemma C.38 tells us that $c(A \wedge B) = c(A)$.

$$\begin{aligned}(T + A)/B &= t(c(A) \cap |A|)/B \\ &= t(c(A) \cap |A| \cap |B|) \\ &= t(c(A) \cap |A \wedge B|) \\ &= T + A \wedge B.\end{aligned}$$

□

C.5.3 Completeness

Now we come to the point in GROVE's paper which prompted this paper. How to get a system of spheres out of each revision function $+$? This paper adopts the approach of constructing a system of spheres out of the entrenchment relation.

Suppose that we have any entrenchment relation $\leqslant$ on F which starts at T.

Definition C.41. For $A \in F$ define

$$s(A) = \{x \in M_L \mid x \text{ is } A\text{-consistent}\}$$

and set $S_\leqslant = \{s(A) \mid A \in F\}$.

Naturally enough, this is a system of spheres, but before we prove this we deduce a few preliminary results on how closely this system tracks the underlying entrenchment relation.

Lemma C.42. $(\forall A, B \in F)\,[s(A) \subseteq s(B) \iff A \leqslant B]$.

Proof. Let $A, B \in F$. Firstly if $B \notin K$ then

$$\begin{aligned} s(B) &= \{x \in M_L \mid x \text{ is } B\text{-consistent}\} \\ &= \{x \in M_L \mid x \text{ is consistent}\} \\ &= M_L, \end{aligned}$$

so $s(A) \subseteq M_L = s(B)$ and $A \leqslant B$ by definition. So take $B \notin K$,

($\Longleftarrow$) Suppose that $A \leqslant B$ and let x be A-consistent and so by Proposition C.11 it is B-consistent.

($\Longrightarrow$) Suppose that $s(A) \subseteq s(B)$. Let x be any maximal consistent A-consistent extension of $\{A\}$ whose existence is guaranteed by Lemma C.26. Thus $x \in s(A) \subseteq s(B)$, but $A \in x \in s(B)$ so $A \leqslant B$.

□

Lemma C.43. *For $A \in K$, $s(A) \cap |A| \neq \varnothing$.*

Proof. Let x be a maximal consistent A-consistent extension $\{A\}$. Thus $A \in x$ and $x \in s(A)$, i.e., $x \in |A| \cap s(A)$. □

Lemma C.44. *For all $A, B \in F$, $s(B) \cap |A| \neq \varnothing \Longrightarrow A \leqslant B$.*

Proof. Let $x \in s(B) \cap |A|$. Thus $A \in x$ and x is B-consistent, hence $A \leqslant B$ by Definitions C.8 and C.10. □

Theorem C.45. *The collection* $\mathbf{S}_{\leqslant}$*, as defined, is a system of spheres centered on* $|T|$.

Proof. We proceed by showing that all the sphere conditions hold for $\mathbf{S}$.

(S1) $\mathbf{S}_{\leqslant}$ is totally ordered by $\subseteq$.

Let $A, B \in F$. Thus either $A \leqslant B$ or $B \leqslant A$ by condition ($\leqslant$1) and hence $s(A) \subseteq s(B)$ or $s(B) \subseteq s(A)$ by Lemma C.42.

(S2) $|T|$ is the $\subseteq$-minimum of $\mathbf{S}_{\leqslant}$.

Let $A \in T$. Thus $\sim A \notin T$ so A is the $\leqslant$-minimum of F. Hence by Lemma C.42 $s(A)$ is the $\subseteq$-minimum of $\mathbf{S}_{\leqslant}$. Now,

$$\begin{aligned} s(A) &= \{x \in M_L \mid x \text{ is } A\text{-consistent}\} \\ &= \{x \in M_L \mid (\forall B \in x)\,[B \leqslant A]\} \\ &= \{x \in M_L \mid (\forall B \in x)\,[\sim B \notin T]\} \\ &= \{x \in M_L \mid T \subseteq x\} \\ &= |T|, \end{aligned}$$

where the second line follows because $\mathrm{Cn}\,(x) = x$, the third by A being a $\leqslant$-minimum, and the fourth line since each $x \in M_L$ is maximal consistent.

(S3) $M_L \in \mathbf{S}_{\leqslant}$.

$$\begin{aligned} M_L &= \{x \in M_L \mid x \text{ is consistent}\} \\ &= \{x \in M_L \mid x \text{ is } \perp\text{-consistent}\} \\ &= s(\perp) \in \mathbf{S}_{\leqslant}. \end{aligned}$$

(S4) The Minimality Condition.

Let $A \in F$ and suppose that $s(A) \cap |A| \neq \varnothing$. Thus $A \in K$.

Let $S \in \mathbf{S}_{\leqslant}$ and suppose that $S \subsetneq s(A)$. Thus $S = s(B) \subsetneq s(A)$ for some B. Hence $B \leqslant A$ and $A \nleqslant B$, so by Lemma C.44 $s(B) \cap |A| = \varnothing$.

□

Before we move on, note from the proof of the last theorem that in $\mathbf{S}_{\leqslant}$, the system of spheres induced by a relation $\leqslant$, $c(A) = s(A)$ for each $A \in F$.

What we just produced was a method for finding a system of spheres from any entrenchment relation $\leqslant$. Now suppose that we restrict ourselves to the $\leqslant_+$ defined as per Definition C.30 from a revision function $+$ and take $\mathbf{S}_+$ to be the system of spheres induced by $\leqslant_+$ (in our earlier notation that would be $\mathsf{S}_{\leqslant_+}$). We are then required to verify that the revision function induced by this system of spheres is precisely the original revision function that we started with.

Theorem C.46. $+_{\mathsf{S}_+} = +$.

Proof. Let $A \in F$, we need to show that
$\underline{T + A = T +_{\mathsf{S}_+} A.}$

So let $B \in F$.[11]

$$\begin{aligned}
B \notin T +_{\mathsf{S}_+} A &\iff B \notin t(c(A) \cap |A|) \\
&\iff B \notin t(s(A) \cap |A|) \\
&\iff (\exists x \in s(A) \cap |A|)\,[B \notin x] \\
&\iff s(A) \cap |A| \cap |{\sim}B| \neq \varnothing \\
&\iff s(A) \cap |A \wedge {\sim}B| \neq \varnothing \\
&\iff A \wedge {\sim}B \leqslant A \\
&\iff T + (A \wedge {\sim}B) \vee A \subseteq T + A \wedge {\sim}B \\
&\iff T + A \subseteq T + A \wedge {\sim}B \\
&\iff {\sim}B \text{ is consistent with } T + A \\
&\iff B \notin T + A.
\end{aligned}$$

□

C.5.4 Correspondence Between Spheres and Entrenchment Relations

We have already seen that if $\leqslant$ is an entrenchment relation starting at $|T|$ then $\mathsf{S}_{\leqslant}$ is a system of spheres centered on $|T|$. As GROVE mentions [36, p. 164]:

Definition C.47. If we are to have a system of spheres S centered on $|T|$ then we can define an entrenchment $\leqslant_{\mathsf{S}}$ by

$$A \leqslant_{\mathsf{S}} B \iff c(A) \subseteq c(B).$$

Theorem C.48. *The relation $\leqslant_{\mathsf{S}}$, as defined, is almost an entrenchment relation starting at T, as it satisfies ($\leqslant$1), ($\leqslant$2), ($\leqslant$3), and ($\leqslant$4).*

[11] Note that this proof, in some sense, extends the proof of Theorem C.34.

Proof. We verify points ($\leqslant$1) through ($\leqslant$4).

($\leqslant$1) $(\forall A, B \in F)\,[A \leqslant_S B$ or $B \leqslant_S A]$.

Let $A, B \in F$. Either $c(A) \subseteq c(B)$ or $c(B) \subseteq c(A)$ by the linearity of **S**. Thus $A \leqslant_S B$ or $B \leqslant_S A$.

($\leqslant$2) $(\forall A, B, C \in F)\,[A \leqslant_S B$ and $B \leqslant_S C \Longrightarrow A \leqslant_S C]$.

Let $A, B, C \in F$ and suppose that $A \leqslant_S B$ and $B \leqslant_S C$. Then $c(A) \subset c(B)$ and that $c(B) \subseteq c(C)$ and so $c(A) \subseteq c(C)$, so $A \leqslant_S C$.

($\leqslant$3) $(\forall A, B, C \in F)\,[A \to B \vee C \in L \Longrightarrow B \leqslant_S A$ or $C \leqslant_S A]$.

Suppose that $A, B, C \in F$ and that $A \to B \vee C \in L$. Thus $|A| \subseteq |B \vee C|$. Hence

$$\varnothing \neq c(A) \cap |A| \subseteq c(A) \cap |B \vee C|,$$

so one of

$$c(A) \cap |B| \neq \varnothing \text{ or } c(A) \cap |C| \neq \varnothing,$$

hence by minimality

$$c(B) \subseteq c(A) \text{ or } c(C) \subseteq c(A),$$

and so $B \leqslant_S A$ or $C \leqslant_S A$.

($\leqslant$4) $(\forall A \in F)\,[(\forall B \in F)\,[A \leqslant_S B] \Longleftrightarrow \sim A \notin T]$.

Let $A \in F$. We prove each direction separately.

($\Longrightarrow$) Suppose that $\sim A \in T$. Thus

$$|A| \cap c(\top) = |A| \cap |T| \subseteq |A| \cap |\sim A| = \varnothing.$$

which tells us that $c(\top) \subsetneq c(A)$ and so $c(A) \nsubseteq c(\top)$. Thus $A \nleqslant_S \top$.

($\Longleftarrow$) Suppose that $\sim A \notin T$, so $|A| \cap |T| \neq \varnothing$. Hence $c(A) \subseteq |T|$, the $\subseteq$-minimum of **S**, so $c(A) = |T| \subseteq c(B)$ for all $B \in F$. Thus $(\forall B \in F)\,[A \nleqslant_S B]$.

□

However, GROVE is incorrect when he says that $\leqslant_S$ satisfies all the entrenchment conditions, as the following example shows.

Example C.49. Let $D \in K$, with $\sim D \in K$, and let $\mathrm{S} = \{|D|, M_L\}$. Thus S is trivially a system of spheres centered on $\mathrm{Cn}(\{D\})$, however

$$\begin{aligned} \sim D \text{ is } \leqslant_{\mathrm{S}}\text{-maximal} &\iff c(\sim D) = M_L \\ &\iff |\sim D| \cap |D| = \varnothing. \end{aligned}$$

This last line trivially holds, so $\sim D$ is $\leqslant_{\mathrm{S}}$-maximal, but $\sim\sim D \notin L$, so $\leqslant_{\mathrm{S}}$ does not satisfy ($\leqslant$5).

This is really only a technical complaint, an artifact of our formalism which requires that inconsistent formulae be explicitly taken account of in the $\leqslant$-relation, whereas they are taken care of in the background by the system of spheres formalism.

The operation of moving from a system of spheres to a $\leqslant$-relation and then back again does not necessarily get you back to where you started as we will see in the Section C.6 when we show that $\mathrm{S}_{\leqslant}$ is always closed yet an arbitrary system of spheres need not be.

However, the other full circle, namely that of starting with an entrenchment, moving to a system of spheres and then returning to an entrenchment, does complete.

Theorem C.50. *Let $\leqslant$ be an entrenchment relation starting at $|T|$ and let $\mathrm{S}_{\leqslant}$ be the system of spheres induced by $\leqslant$. Then $\leqslant_{\mathrm{S}_{\leqslant}} = \leqslant$.*

Proof. Let $A, B \in F$. Then

$$\begin{aligned} A \leqslant_{\mathrm{S}_{\leqslant}} B &\iff c(A) \subseteq c(B) \\ &\iff s(A) \subseteq s(B) \\ &\iff A \leqslant B, \end{aligned}$$

where this last line is just Lemma C.42. □

C.6 Closed Systems of Spheres

Certainly it is nice that we have a formalism that is able to deal with systems of spheres which can have arbitrary makeup, but it would be equally nice if, when we so choose, we could work with "closed" or "elementary" systems of spheres. That is when each sphere within the system is a closed set within the *Stone Topology* or more revealingly when each sphere is $|T'|$ for some theory T'.

C.6.1 Some Basic Topology

Since we will be using the terminology of basic point set topology here and some results well known to topology students may not be well known to those who study belief revision, it will be worth going over some basic results about the topology of our Stone spaces and, in particular, those to do with closures.

Definition C.51. The *Stone Topology* on M_L is the topology with basis

$$\{|A| : A \in F\}.$$

This means that the open sets of this topology are those sets which are arbitrary unions of $|A|$s. Note that the collection of open sets, as well as the basis, are closed under finite intersections and unions. The collection of open sets is trivially closed under arbitrary unions.

Definition C.52. A set $U \subseteq M_L$ is *closed* iff $M_L - U$ is open.

Notice that each set $|A|$ is closed as $M_L - |A| = |{\sim}A|$ is open. Since it is both open and closed, we refer to this set as being *clopen*. Note also that the collection of closed sets is closed under finite unions and arbitrary intersections.

Proposition C.53. *A set $U \subseteq M_L$ is closed iff $U = |\Sigma|$ for some set Σ; and Σ can be taken to be in $\mathbb{T}$.*

Proof. Suppose that U is closed in M_L. Thus

$$U = M_L - \bigcup_{i \in I} |A_i|,$$

for some arbitrary collection of formulae $\Sigma = \{{\sim}A_i \mid i \in I\} \subseteq F$. Hence by the De Morgan laws,

$$U = \bigcap_{i \in I} (M_L - |A_i|),$$

so by $|\cdot|$ properties,

$$U = \bigcap_{i \in I} |{\sim}A_i| = |\Sigma|.$$

We then have that U is of the desired form, and if we put $T' = t(|\Sigma|)$ we then have that $|\Sigma| = |T'|$, so we might as well have taken $\Sigma \in \mathbb{T}$ to start with. □

With 'closed sets' comes the notion of 'closure':

Definition C.54. The *closure*, $\mathrm{cl}(X)$, of a set $X \subseteq M_L$ is defined to be the smallest closed set in M_L which contains X.

Since the collection of closed sets is closed under arbitrary intersection and since M_L is closed ($M_L = M_L - \varnothing = M_L - |\bot|$), the notion of closure is well defined. Note that the closure of a closed set is, naturally, closed.

A more useful test for being in the closure of X is the following:

Proposition C.55. *An element $x \in M_L$ is in $\mathrm{cl}(X)$ for a set $X \subseteq M_L$ iff*

$$(\forall A \in F)\,[x \in |A| \Longrightarrow |A| \cap X \neq \varnothing]\,.$$

This alternate definition says that x is in the closure of X if no matter what neighborhood we take around x, that neighborhood will intersect X.

Proof. Let $x \in M_L$, and let $X \subseteq M_L$.

($\Longrightarrow$) Suppose that it is not true that

$$(\forall A \in F)\,[x \in |A| \Longrightarrow |A| \cap X \neq \varnothing]$$

That is

$$(\exists A \in F)\,[x \in |A| \text{ and } |A| \cap X = \varnothing]\,.$$

Thus $X \subseteq |{\sim}A|$ and $x \notin |{\sim}A|$. Since $|{\sim}A|$ is a closed set larger than X and since $x \notin |{\sim}A|$ we know that $\mathrm{cl}(X)$ is smaller than $|{\sim}A|$ and so $x \notin \mathrm{cl}(X)$.

($\Longleftarrow$) Suppose that $x \notin \mathrm{cl}(X)$. Thus x is in the open set $M_L - \mathrm{cl}(X)$. Hence by the definition of open sets there is some $A \in F$ such that $x \in |A| \subseteq M_L - \mathrm{cl}(X)$. So, $x \in |A|$ and

$$|A| \cap X \subseteq |A| \cap \mathrm{cl}(X) = \varnothing.$$

Thus establishing that

$$(\forall A \in F)\,[x \in |A| \Longrightarrow |A| \cap X \neq \varnothing]$$

does not hold.

□

C.6.2 Systems of Spheres

With this small smattering of definitions and results from elementary topology we are ready to proceed with our discussion on closed systems of spheres.

Definition C.56. A system of spheres S is said to be *closed* iff

$$(\forall S \in \mathbf{S})\,[S \text{ is closed in } M_L]\,.$$

If S is a system of spheres, then its closure is defined to be

$$\mathrm{cl}\,(\mathbf{S}) = \{\mathrm{cl}\,(S) \mid S \in \mathbf{S}\}\,.$$

Note that a system of spheres is closed iff each of its constituent spheres represents a full theory; that is elementary in the sense of Subsection C.5.2.

Let us note that not all systems of spheres are closed:

Example C.57. Let x_0 be some point in M_L. Then $\mathbf{S} = \{M_L - \{x_0\}\,, M_L\}$ is a system of spheres which is not closed as clearly $\mathrm{cl}\,(M_L - \{x_0\}) = M_L$.

Our next observation is that the systems of spheres we constructed out of entrenchment relations are nice and elementary/closed.

Theorem C.58. *Suppose that $\leqslant$ is an entrenchment relation. Then*

$$\mathbf{S}_{\leqslant} = \{s(A) \mid A \in F\}$$

defined as in Subsection C.5.3 is closed.

Proof. Let $A \in F$. We will show that $\mathrm{cl}\,(s(A)) \subseteq s(A)$ (since, trivially, $s(A) \subseteq \mathrm{cl}\,(s(A))$, this gives us our result). So let $x \notin s(A)$. Thus x is not A-consistent, which means that there is a $B \in x$ such that $B \nleqslant A$.

$\underline{|B| \cap s(A) = \varnothing.}$

Assume that there exists a $y \in |B| \cap s(A)$. Hence $B \in y$ and y is A-consistent, so $B \leqslant A$, a contradiction.

Hence $x \in |B|$ and $|B| \cap s(A) = \varnothing$, so $x \notin \mathrm{cl}\,(s(A))$. □

Remark C.59. Here we have another demonstration that the operation of converting a system of spheres into an entrenchment relation is not reversible, since any, possibly non-closed, system of spheres will give rise to an entrenchment relation which will, in turn, give rise to a closed system of spheres.

Another way of seeing this result would have been the cruder way of looking at the cardinalities involved. There are at most the cardinality of $\mathcal{P}\,(F \times F)$ $(= 2^{\omega})$ many entrenchment relations whereas there are at least the cardinality of $\{f : 2^{\omega} \longrightarrow M_L \mid f \text{ is 1-1}\}$ $(= 2^{2^{\omega}})$ many systems of spheres, so something must be lost when we move systems of spheres to entrenchments.

We thus have a method of taking systems of spheres and deriving closed systems of spheres from them: From S, form $+_{\mathsf{S}}$, and from this form $\mathsf{S}_{+_{\mathsf{S}}}$.[12] Theorem C.46 then guarantees that this new, closed, system of spheres is equivalent to our original system of spheres in the sense that $+_{\mathsf{S}_{+_{\mathsf{S}}}} = +_{\mathsf{S}}$.

Actually though, we can strengthen the appearance of this result to state that $\mathrm{cl}(\mathsf{S})$ is equivalent to S, and this will be the last result of this appendix. To get there, however, we will need to verify that $\mathrm{cl}(\mathsf{S})$ really is a system of spheres and we must prove a small technical Lemma.

Lemma C.60. *Suppose that* S *is a system of spheres centered on* X. *Then* $\mathrm{cl}(S)$ *is a system of spheres centered on* $\mathrm{cl}(X)$.

Proof. We verify each of the sphere conditions.

(S1) $\mathrm{cl}(\mathsf{S})$ is linearly ordered by $\subseteq$.

Suppose that $\mathrm{cl}(U), \mathrm{cl}(V) \in \mathrm{cl}(\mathsf{S})$ for some $U, V \in \mathsf{S}$. Without loss of generality take $U \subseteq V$. Then $\mathrm{cl}(U) \subseteq \mathrm{cl}(V)$ by closure properties.

(S2) $\mathrm{cl}(X)$ is the $\subseteq$-minimum of $\mathrm{cl}(\mathsf{S})$.

We know that $X \in \mathsf{S}$, so $\mathrm{cl}(X) \in \mathrm{cl}(\mathsf{S})$. Suppose that $\mathrm{cl}(U) \in \mathrm{cl}(\mathsf{S})$ for some $U \in \mathsf{S}$. Thus $X \subseteq U$ so $\mathrm{cl}(X) \subseteq \mathrm{cl}(U)$.

(S3) $M_L \in \mathrm{cl}(\mathsf{S})$.

$M_L \in \mathsf{S}$, so $M_L = \mathrm{cl}(M_L) \in \mathrm{cl}(\mathsf{S})$.

(S4) The Minimality Condition.

Suppose that $A \in F$ and further suppose that $\mathrm{cl}(U) \cap |A| \neq \varnothing$, so there exists an $x \in \mathrm{cl}(U) \cap |A|$. Since $|A|$ is an open neighborhood of x, and $x \in \mathrm{cl}(U)$, by Proposition C.55, $U \cap |A| \neq \varnothing$. So by The Minimality Condition in S we have that there is a minimal U_0 such that $U_0 \cap |A| \neq \varnothing$.

Claim: $\mathrm{cl}(U_0)$ is minimal in $\mathrm{cl}(\mathsf{S})$ such that $\mathrm{cl}(U_0) \cap |A| \neq \varnothing$.

Proof of claim:

Suppose that $V \in \mathsf{S}$ such that $\mathrm{cl}(V) \subseteq \mathrm{cl}(U_0)$ and $\mathrm{cl}(V) \cap |A| \neq \varnothing$. Thus, $V \cap |A| \neq \varnothing$, and so $U_0 \subseteq V$ by The Minimality Condition. Thus $\mathrm{cl}(U_0) \subseteq \mathrm{cl}(V)$, telling us that $\mathrm{cl}(U_0) = \mathrm{cl}(V)$, as required. *End of proof of claim.*

□

[12] Note that $\mathsf{S}_{+_{\mathsf{S}}}$ is really $\mathsf{S}_{\leqslant_{+_{\mathsf{S}}}}$ and so is a closed system of spheres.

Lemma C.61. *Let* S *be a system of spheres centered on* X. *Then for all* $A \in K$,

$$\mathrm{cl}\,(c_{\mathbf{S}}(A)) = c_{\mathrm{cl}(\mathbf{S})}(A),$$

where $c_{\mathbf{S}}$ *is the Minimal Intersection Function for the system of spheres* $\mathbf{S}$, *and* $c_{\mathrm{cl}(\mathbf{S})}$ *is the Minimal Intersection Function for the system of spheres* $\mathrm{cl}\,(\mathbf{S})$.

Proof. Let $A \in K$. Thus by definition of $c_{\mathbf{S}}$, $c_{\mathbf{S}}(A) \cap |A| \neq \varnothing$, so $\mathrm{cl}\,(c_{\mathbf{S}}(A)) \cap |A| \neq \varnothing$, and we can conclude that

$$c_{\mathrm{cl}(\mathbf{S})}(A) \subseteq \mathrm{cl}\,(c_{\mathbf{S}}(A))\,.$$

There is a $U \in \mathbf{S}$ such that $c_{\mathrm{cl}(\mathbf{S})}(A) = \mathrm{cl}\,(U)$, so $\mathrm{cl}\,(U) \cap |A| \neq \varnothing$, so by the same reasoning as in the proof of (S4) in Lemma C.60, we have that $U \cap |A| \neq \varnothing$, so $c_{\mathbf{S}}(A) \subseteq U$. Hence

$$\mathrm{cl}\,(c_{\mathbf{S}}(A)) \subseteq \mathrm{cl}\,(U) = c_{\mathrm{cl}(\mathbf{S})}(A),$$

and together with the above inclusion we can then conclude equality. □

Theorem C.62. *Suppose that* $\mathbf{S}$ *is a system of spheres centered on* $|T|$. *Then*

$$+_{\mathbf{S}} = +_{\mathrm{cl}(\mathbf{S})}\,.$$

Proof. Firstly note that by Lemma C.60, $\mathrm{cl}\,(\mathbf{S})$ is a system of spheres centered on $|T|$ (since $|T| \in \mathbf{S}$, so $\mathrm{cl}\,(|T|) = |T| \in \mathrm{cl}\,(\mathbf{S})$).

Let $A \in F$. We want to show that $T +_{\mathbf{S}} A = T +_{\mathrm{cl}(\mathbf{S})} A$, i.e.,

$$(\forall B \in F)\left[B \in T +_{\mathbf{S}} A \Longleftrightarrow B \in T +_{\mathrm{cl}(\mathbf{S})} A\right].$$

So let $B \in F$, and consider $c_{\mathbf{S}}(A)$ and $c_{\mathrm{cl}(\mathbf{S})}(A) = \mathrm{cl}\,(c_{\mathbf{S}}(A))$. We then get the following sequence of equivalences:

$$\begin{array}{rcl}
\mathrm{cl}\,(c_{\mathbf{S}}(A)) \cap |A \wedge {\sim}B| = \varnothing & \Longleftrightarrow & c_{\mathbf{S}}(A) \cap |A \wedge {\sim}B| = \varnothing \\
 & \Downarrow & \\
\mathrm{cl}\,(c_{\mathbf{S}}(A)) \cap |A| \cap |{\sim}B| = \varnothing & \Longleftrightarrow & c_{\mathbf{S}}(A) \cap |A| \cap |{\sim}B| = \varnothing \\
 & \Downarrow & \\
\mathrm{cl}\,(c_{\mathbf{S}}(A)) \cap |A| \subseteq |B| & \Longleftrightarrow & c_{\mathbf{S}}(A) \cap |A| \subseteq |B| \\
 & \Downarrow & \\
B \in t\,(\mathrm{cl}\,(c_{\mathbf{S}}(A)) \cap |A|) & \Longleftrightarrow & B \in t\,(c_{\mathbf{S}}(A) \cap |A|) \\
 & \Downarrow & \\
B \in T +_{\mathrm{cl}(\mathbf{S})} A & \Longleftrightarrow & B \in T +_{\mathbf{S}} A
\end{array}$$

The first line is true by closure properties and so the last line obtains. □

C.7 Conclusion

This paper has revisited GROVE's result [36] in close detail and through this detail we have seen the usefulness of an epistemic entrenchment as our starting point rather than systems of spheres. Also we have provided a new way of looking at the revision induced by an epistemic entrenchment and it is the author's hope that this will further enhance our understanding of revision operations.

Appendix D

Combining Finite Model Generation with Theorem Proving: *Problems and Prospects*

With

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ABSTRACT: This appendix is about automated proof search, automatic searching for models and the potentially fruitful ways in which these traditionally separate aspects of reasoning may be made to interact. It takes its starting point in research reported in 1993 (Slaney, *SCOTT: A Semantically Guided Theorem Prover*, Proc. 13th IJCAI) on a system which combines a high performance first order theorem prover with a program generating small models of first order theories. The main theorem is an incompleteness result for a certain range of problems to which this combined system has been successfully applied. While the result may not be unexpected, the proof is worth examining and it is important to reflect on its relationship to the research program in combining methods.

D.1 Proof search, model search and their interaction

D.1.1 Background

Traditional theorem provers search without much intelligence. They may reason forwards from the axioms or backwards from the desired theorem or both, but in either direction they rapidly find themselves in an exponentially growing search space of possible proof fragments which they explore in a manner at once admirably industrious and remarkably dull. Much good work in automatic proof search has gone into the discovery and refinement of methods for controlling the explosion of the search space or for reducing the amount of duplicated work undertaken in traversing it. At the same time, there has been vigorous research on heuristics for directing the search for proofs, but many of the most successful ideas in that field are either unexciting suggestions such as exploring short formulae first or else mysterious ones such as preferring some operator to be nested to the left rather than to the right.[1] The root problem is that most powerful theorem provers work only locally, focussing on the specific formulae being transformed by an inference rather than on global aspects of the situation, and more significantly they are based on pure syntax.[2] They may take into account questions like how many function symbols a formula has, which is the first literal in a clause, whether this unifies with that and the like, but they do not consider what the inference step is supposed to achieve, whether it is establishing a general law for the structures under consideration or an accidental property of the case, whether the conclusion of the inference says the same thing as one already proved or the like. That is, traditional provers do not understand what they are doing.

Still less do they understand the problems they are attempting to solve. The contrast with human theorem provers is striking. When we reason, we appeal constantly to conceptual structures within which particular problems make sense. We are not capable of exploring search spaces of millions of clauses, and it seems that we do not need to. What we are able to use is some sense of when a proof search is getting closer to the goal. What lies behind this capacity is not so much an ability to recognise syntactic patterns in formulae (though we importantly have that too) as an understanding of the problem: we know what the symbols, theories, axioms, goals and subgoals *mean*, and this puts us at an advantage in the investigation.

These remarks may be taken as leading to a recommendation that researchers in automated reasoning direct their efforts towards mechanical emulation of human cognitive processes. This, however, would be a mistake, certainly in the present state of development of the discipline. In the details of what they do—in what they find easy and what hard—computers are quite unlike us. Without becoming totally discouraged from projects in programming naturalistic intelligence, we should recognise that for practical purposes the indicated way ahead is to continue to let the machines be mechanical, doing

[1]Not all research in the area has been like that. The present appendix is part of a minority tradition characterised by attempts to direct semantic reasoning for syntactic purposes. Plaisted and Caferra should be mentioned as important recent contributors to this tradition, whose roots go back to very early work by Gelernter and others. The present point is just that this is a *minority* tradition.

[2]This is not to deny a semantic basis for the usual rules and methods, but at the level of the individual inference step everything is most easily characterised in syntactic terms.

what they do best at the high speeds of which they are capable.[3] Algorithmic proof search is good for some things anyway; it no more needs pseudo-human cognition in order to out-infer a mathematician than a car needs legs to outrun an athlete.

So we do not know how to program understanding, and in any case attempting to do so is likely to lead to inefficient systems. What we may rather hope to do is to secure some of the useful effects of understanding in a way congenial to algorithmic processes. To understand something is to know what it means. While that may be unattainable by means of software of the kind we know how to build, it can rather simply be *approximated*. It remains to be seen whether such approximations really pay their way, though the initial indications are good.

What is computationally possible is to *interpret* a formal language as referring to objects in a particular domain of discourse. An interpretation in this sense is just what we were all told it was when, at our mother's knee, we were introduced to model theory. That is, it consists of a nonempty set called the domain and a function which assigns to each predicate symbol a relation of appropriate arity defined on the domain and to each function symbol a function similarly. Elaborations to allow for possible worlds, impossible worlds, higher order structures, multiple sorts or whatever may be pasted into this framework as desired. With an interpretation is associated a notion of truth, fleshed out by the familiar inductive conditions for evaluating molecular formulae in terms of the values of their parts. Thus an interpretation—any interpretation—divides the language into the true and the false. That is, it marks a difference in meaning between half of what may be asserted and the other half. Even a crude interpretation is rich in semantic information of the kind which human reasoners may glean from their understanding and which may help automatic reasoning systems behave more intelligently.

D.1.2 Uses of interpretations

The oldest and simplest way of using an interpretation to help guide the search for a proof is to delete unprovable subgoals. In searching backwards from a goal, the prover typically looks at the available rules of inference and asks how they may have been applied to generate this particular goal (theorem) from subgoals (lemmata). Sometimes the rules are invertible, so that if the goal is provable then so are the subgoals, but this is not always the case. For

[3] And of course to facilitate the right kind of human-machine interactions. In order to secure the insights of a mathematician, it is more effective to plug in a mathematician at some appropriate point than to try to emulate one in an unsuitable medium like that provided by current computer technology.

example, if the goal is $\exists x A$ then *one* way to derive it would be to infer it from some particular A_t, so one possible subgoal would be to derive A_t. However, it is possible that A_t does not follow from the axioms of the problem even if $\exists x A$ does. If we have an interpretation which is a model of the theory in which the proof is sought (that is, of the axioms of the problem) then we may detect the unprovability of some subgoals by testing them against the model and finding that they are false. Then there is no point in trying to derive them, so the search can immediately backtrack and try another subgoal. Diagrams in geometry have this function among others: we do not waste time trying to prove two triangles congruent if it is obvious from the diagram that they have different shapes.

In the case of backward reasoning, there is no danger that the formulae considered will be irrelevant to the goal, but there is a danger that they will not follow from the assumptions. In the case of forward reasoning, the problem is the opposite one: all of the formulae considered are derivable, but most of them play no part in proofs of the target theorem. Here there is another use for interpretations. Given an interpretation, or a set of them, in which the goal is *false*, it makes sense to explore most vigorously the consequences of axioms and lemmata which are also false. This *false preference strategy* rests on the thought enunciated above, that interpretations reveal something of the semantic character of formulae. In the context of the question "How does this set of formulae entail that conclusion?" it corresponds to saying "Here is a way for the conclusion to be false, and *those* formulae fail with it, so concentrate on them."

Another strategy, most germane to the technical part of this appendix, is semantic restriction of rules of inference. Given an interpretation—usually again one in which the target theorem is false—the rules of inference may be barred from applying unless one of the premises of the inference is false in the interpretation. Sometimes, as in the case of resolution as a refutation procedure with the empty clause as goal, this kind of restriction is complete in that if there is a derivation at all there is one obeying the model-based condition. In other cases it is incomplete and so may cause proof searches to fail where they would otherwise have succeeded. In such cases it may be weakened to become a variety of false preference, assigning weights or the like not to prohibit the inferences which violate it but to delay them or render them less likely to be selected.

D.1.3 Dynamic model generation

It may be that some particular interpretation is known to be apposite for a class of problems, in which case it may be given to a theorem prover along with the problem definition or even hard-coded into the prover itself. However, in many cases we do not know in advance what interpretation will be useful. One option is then to use an all-purpose interpretation, perhaps selected to be such as to make testing for truth value very fast. For example, ordinary hyper-resolution uses an interpretation in which all atoms are false. It has long been observed [76] that such an interpretation is unlikely to be ideal for any particular problem, and that an important project in automated reasoning is to find ways of suiting interpretations automatically to problems. Another option, therefore, is to combine a theorem prover with a program which searches for models of theories. This should be done in such a way that:

1. The models generated are adapted to the proof search in hand.

2. Testing formulae in the models is fast.

3. Searching for models does not occupy more time than is saved by using them to provide semantic information.

4. Replacing models by "better" ones as the proof search progresses does not compromise the soundness or completeness of the prover.

Item 1 means that, for example, in a case of goal deletion the interpretations should falsify a lot of the subgoals encountered in the present proof search while remaining models of the axioms. In the case of forward reasoning strategies, the models should not only falsify the goal but make a high proportion of the derived formulae true so that they focus the search significantly. Items 2 and 3 are obvious given that it is a strategic error to put more resources into any strategy than are covered by the return from it. Item 4 is an integrity condition.

One (generic) system capable of meeting these conditions consists of three modules:

Prover. Some form of theorem prover which takes as input a problem (axioms, rules, goal, ...) and gives as output either a proof or failure. It can use information as to whether individual formulae are true or false in some model, but it knows nothing about the content of the model.

Modeller. A program which takes as input two theories, a background theory about the domain of investigation and a set of formulae to be interpreted, together with conditions which limit its search to force termination, and gives as output a model of the theories if it finds one, or a failure message otherwise. It uses formulae from the prover as axioms, but knows nothing about inference rules or proof strategy.

Tester. A module which mediates between the other two. The prover may send a formula to the tester and get back the formula's truth value. The tester may send a theory to the modeller and get back either a model of that theory or failure if there is no model in the search space.

The entire system is in one of two states: active or passive. In passive mode, the modeller is inactive; the tester has an interpretation in memory and it merely tests formulae in that and returns the results. In active mode the tester maintains not only the current interpretation M but also a current theory T, of which M is a model. During the proof search, a series of formulae arrive at the tester to be assigned truth values. With each formula A the tester does:

```
If A is false in M then
    Call the modeller with theory T ∪ {A}
    If a model N is returned then
        M ← N
    Else return FALSE
Endif
T ← T ∪ {A}
Return TRUE
```

A 'theory' here is just a set of formulae. Initially, T is null and the initial M is obtained by calling the modeller with the null theory, thus getting a model of the background theory (which may also be null, in which case some dummy model is returned). As a result of all this, after a while the tester's theory T consists of formulae which have occurred during the proof search and which have been evaluated as true, and its interpretation M has been generated specifically to make true as many as possible of the formulae from the prover. Thus M is not arbitrary, but is adapted to the particular problem being addressed by the prover and to the particular proof strategy used.

The theory T is there in order to comply with item 4 on the list of desiderata. It ensures that once the prover has been told that a formula is true, that formula remains true even when the guiding model is changed.

Because searching for models is computationally expensive compared with checking a formula against a given model, the system should switch at some point from active to passive mode. This should happen when it is likely that the current model is as good as any within the search space. Of course, the ways of recognising when that point is reached are likely to be fallible, but then heuristics generally are fallible. A simple way would be to stop generating after a certain pre-defined number of formulae have been tested, or alternatively after a given number of consecutive failures to find models.

The generic combined system has been implemented [77,79]. The program SCOTT combines the theorem prover Otter [62] with the model generator FINDER [78].[4] It is worth remarking that FINDER searches for *small* models, so usually testing formulae against them is fast compared with generating and processing consequences in the prover. The techniques so far implemented are semantic restriction of rules and false preference. The latter is effected by assigning weight to each true clause, allowing Otter's normal strategy of choosing lightest clauses first to do the rest.

SCOTT must be regarded as a preliminary essay in combining semantic and syntactic methods. Nonetheless, it exhibits some of the pleasing features one might expect from such a combination. Otter is powerful and fast, but it searches without much intelligence. Its technique is to spray out consequences in all directions, remove some dross (such as subsumed clauses) and hope. FINDER is capable of extracting rich information from a few axioms, but it is powerless to deduce consequences. In the combined system, the two inform each other achieving more than either could alone.

D.2 Incompleteness

D.2.1 Condensed detachment

Hilbert systems for propositional logic with an implication connective $\rightarrow$ traditionally take as primitive some axiom schemata and the rule of Modus Ponens or detachment:

$$\frac{A \qquad A \rightarrow B}{B}$$

An interesting variant takes individual formulae (rather than schemata) as axioms and closes under the more general rule of *condensed* detachment—detachment incorporating the substitution required to unify the minor premise with the antecedent of the major one:

$$\frac{C \qquad A \rightarrow B}{B\sigma}$$

where σ is the most general unifier of A and C.

Condensed detachment was introduced by Meredith and received a sustained investigation in the 1980s [47,63]. It is clear that condensed detachment and

[4]The program is available from `ftp://arp.anu.edu.au/pub/SCOTT/` and comes with both Otter's and FINDER's sources. Since the system is described in the cited papers, we do not repeat the account of it here.

resolution are closely related rules, so interest attaches to the question of whether proof search strategies and heuristics similar to those usual for resolution can be adapted to the case of condensed detachment.

Condensed detachment is also well known in the theorem proving community as a source of maddeningly hard problems for classical first order systems. It is easy to represent the formulae of a propositional logic as first order terms, the connectives being function symbols, and to add a unary predicate p for '... is provable'. The rule of detachment goes over into a clause

$$\sim p(x) \vee \sim p(i(x, y)) \vee p(y) .$$

and axioms of the Hilbert systems may simply be asserted as (positive) unit clauses such as

$$p(i(x, i(y, x))) .$$
$$p(i(i(x, i(y, z)), i(i(x, y), i(x, z)))) .$$

To prove a theorem in the propositional logic, Skolemise its negation

$$\sim p(i(i(i(a, b), a), i(i(a, b), b))) .$$

and derive the empty clause in first order logic. Since unification is going to be applied to the first order clauses, the rule of inference of the propositional system is exactly condensed detachment.

Experiments with Otter especially have made the condensed detachment problems into a famous challenge [100].[5] SCOTT has been applied to them with some success [77]. Its performance on such problems is typically better than that of the unaided Otter by a factor of about 2, whether the measure be the number of clauses kept, the number given or the time taken. On certain problems the false preference strategy enables SCOTT to find proofs two orders of magnitude more efficiently than Otter, and in one case SCOTT solved a (minor) open problem in a few seconds after Otter had failed to find a proof in several hours.

It is obvious that where the rule of inference is ordinary binary resolution refutation completeness is not affected, because the tester's theory T ensures that when the guiding model is changed all clauses marked as true remain true in the new interpretation. Where the rule of inference is some other form of semantic resolution such as hyper-resolution, it is equally obvious that further restriction by arbitrary models in the manner of SCOTT may destroy completeness. However, the case of condensed detachment appears to lie somewhere between these two "obvious" cases. In our previous papers [77, 79] the

[5]The challenge of these problems has stimulated other work, for example by the theorem proving groups of ICOT in Tokyo [66] and of the Max Planck Institute in Saarbrücken [35].

question of whether semantically constrained condensed detachment is in any reasonable sense complete was left open.

D.2.2 Definitions

We shall deal with the set of terms over variables $(w, x, y, z, \ldots)$ with a binary function symbol $\rightarrow$ which we write in infix. To save on excessive formalism we also take a term to represent all instances of itself under substitutions which replace distinct variables for distinct variables. We shall denote terms by upper-case Roman letters, $A, B, C, \ldots$, and we shall take Γ to range over the set of terms. Moreover, when dealing with two unconnected terms we assume that they do not share variables.

These terms are analogs of propositional formulae and we can interpret the usual rule of *Modus Ponens* in its most general form as the rule of Condensed Detachment (CD) as stated above, tacitly assuming that $A \rightarrow B$ and C have distinct variables. $\Gamma \vdash^{\mathrm{CD}} A$ then has its usual meaning, i.e. that it is possible to deduce A from the terms in Γ by applications of CD.[6]

Since we have only an attenuated first order language, our semantic structures or *CD-algebras* will be those of a generalised propositional implication, namely algebras $\langle \mathcal{A}, \leqslant, \supset, D \rangle$ where $\mathcal{A}$ is a set on which $\leqslant$ is a binary relation,[7] D is a subset of $\mathcal{A}$ closed under $\leqslant$ and $\supset$ is a binary operation on $\mathcal{A}$ satisfying the condition:

$$a \supset b \in D \iff a \leqslant b$$

The notation $\Gamma \models A$ will then be able to take on its usual meaning and we have a fairly standard (and easy) soundness and completeness result:

$$\Gamma \models B \iff \exists A\,(\exists \sigma (B = A\sigma) \ \wedge \ \Gamma \vdash^{\mathrm{CD}} A)$$

It is here that we pay a small price for the generality imposed on the conclusions of our CD inferences. In general there may not be a CD derivation of a given semantic consequence of Γ, but there will be a derivation of some term which subsumes it.

Semantic constraint by a specific CD-algebra M means that the Condensed Detachment inference from $A \rightarrow B$ and C to $B\sigma$ can proceed only when either

[6]Note that we are not capturing all of the conclusions that would normally follow in a propositional system from the axioms in Γ. This is because we are not allowing arbitrary substitution during our derivations. This issue is interesting from the perspective of logic [63] but not our present concern.

[7]Typically, for interesting logics, $\leqslant$ is a partial order. However, in general it need not be. Our presentation is not maximally efficient, since $\leqslant$ could be defined rather than primitive. We have opted for familiarity rather than economy.

$M \not\models A \rightarrow B$ or $M \not\models C$. If A follows from Γ by M-constrained steps of this form we write $\Gamma \vdash_M A$.

We say that A follows from Γ by Semantically Constrained Condensed Detatchment (SCCD) and write $\Gamma \vdash^{\mathrm{SCCD}} A$ when and only when for every CD-algebra M, if $M \not\models A$ then $\Gamma \vdash_M A$. Then SCCD is complete iff $\Gamma \vdash^{\mathrm{CD}} A$ implies $\Gamma \vdash^{\mathrm{SCCD}} A$. Note that when trying to prove a goal term by SCCD we must use an algebra which invalidates that term. If this restriction were not imposed, there would generally be no semantically constrained proof. At the extreme, we could choose an algebra in which all of the axioms were true and so block all CD inferences from them.

D.2.3 The Counter Example

As promised, we shall give a specific counter example to the conjecture that SCCD is complete. To do this, it will be sufficient to find a set Γ of terms, a conclusion A which follows from Γ by usual CD reasoning, and a CD-algebra M relative to which we can show that A does not follow from Γ by SCCD. That is:

1. $\Gamma \vdash^{\mathrm{CD}} A$,
2. $M \not\models A$, and
3. $\Gamma \not\vdash_M A$.

For our example, we first adopt the notation

$$\begin{aligned} \mathcal{C}_\star(A) &=_{\mathrm{df}} (A \rightarrow v) \rightarrow v \\ \mathcal{K}(A) &=_{\mathrm{df}} v \rightarrow A \end{aligned}$$

where v is a variable (say, the earliest in a standard enumeration) that does not occur in A. Moreover, we write

$$\begin{aligned} \mathcal{C}^0_\star(A) &=_{\mathrm{df}} A \\ \mathcal{C}^{n+1}_\star(A) &=_{\mathrm{df}} \mathcal{C}_\star(\mathcal{C}^n_\star(A)) \end{aligned}$$

Next we define the specific example:

$$\begin{aligned} \alpha &= x \rightarrow \mathcal{C}_\star(x) \\ \beta &= x \rightarrow \mathcal{K}(x) \\ \Gamma &= \{\alpha, \beta\} \\ A &= \mathcal{K}(\mathcal{C}_\star(\alpha)) \end{aligned}$$

Let M be the algebra given by the following matrix:

$$\begin{array}{c|cc} \to & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \qquad \begin{array}{c} D = \{1\} \\ a \leqslant b \iff a = b \end{array}$$

So M interprets $\to$ as the Boolean biconditional, and we can take advantage of the well known fact that a propositional formula containing biconditionals as the only connective is a tautology if and only if the number of occurrences of each propositional variable in that formula is even.

Lemma D.1. *$M \models \alpha$, $M \not\models \beta$, and $M \not\models A$.*

Proof. Count the variable occurrences (*mod* 2). □

Lemma D.2. $\Gamma \vdash^{CD} A$

Proof.

$$\dfrac{\dfrac{\alpha \quad \alpha}{\mathcal{C}_\star(\alpha)} \qquad \beta}{A}$$

□

Thus we are only left with the requirement that $\Gamma \not\vdash_M A$, for which we shall define a set T which includes the M-constrained consequences of Γ, show that T contains only terms of a specific form, and show that A is not of this form. More specifically let T be the set of CD consequences of Γ except that we require that α cannot be applied to itself in the generation of elements of T. This condition corresponds exactly to following the set of support strategy with α as axiom and β initially in the set of support. Note that since α is true in M, all M-constrained derivations satisfy the condition, so clearly

Lemma D.3. *If $\Gamma \vdash_M B$ then $B \in T$.*

It turns out that the converse is also true, but it is not needed for the proof.[8]

To describe the form of the terms in T we shall construct T' (which we will show to be identical to T) as follows:

Definition D.4. T' is the smallest set satisfying:

1. $\alpha \in T'$,

[8] α is the only term in T which is validated by M. This fact is non-trivial but is an easy consequence of Lemma D.6 below.

2. $\mathcal{C}^n_\star(\beta) \in T'$ for all $n \geqslant 0$

3. If $B \in T'$ then $\mathcal{C}^n_\star(\mathcal{K}(B)) \in T'$ for all $n \geqslant 0$

Lemma D.5. $A \notin T'$.

Proof. Assume for reductio that $A \in T'$. Clearly $A \neq \alpha$ and so it was not condition 1 which forced A into T'. Condition 2 does not place A in T' since A is of the wrong form, having a single variable as its antecedent. So $\mathcal{K}(\mathcal{C}_\star(\alpha))$ must be there because of condition 3 with $n = 0$, and therefore $\mathcal{C}_\star(\alpha) \in T'$. However, $\mathcal{C}_\star(\alpha)$ cannot be there in virtue of condition 1 (because $\mathcal{C}_\star(\alpha) \neq \alpha$) or of condition 2 (because $\alpha \neq \beta$) or of condition 3 (because α is not of the form $\mathcal{K}(X)$ since x occurs in $\mathcal{C}_\star(x)$). There being no other way for $\mathcal{C}_\star(\alpha)$ to get into T', this is a contradiction. □

Lemma D.6. $T = T'$.

Proof. The inclusion from right to left can be obtained simply by applying α and β to themselves (excluding, of course, α to α) and then repeatedly applying them to the resultant terms.

For the inclusion in the reverse direction we proceed by induction on the length of the CD proof of B from $\{\alpha, \beta\}$. The case when B is an axiom (i.e. α or β) is straightforward and so we are left to consider what happens when B is the result of applying our restricted CD rule to major premise C and minor premise D given our inductive hypothesis that $C, D \in T'$.

Let us say that a term is of form i ($1 \leqslant i \leqslant 3$) if it is in T' in virtue of condition i of Definition D.4.

There are several simple cases to be disposed of. We deal first with the possibility that C is either α or one of the "degenerate" cases of form 2 or 3 with $n = 0$. Then we deal with the cases in which D is of one of those forms.

1. It is not allowed that both C and D are of form 1.

2. If C is of form 1 (i.e. $C = \alpha$) and D is of form 2 or 3, then B is just $\mathcal{C}_\star(D)$ which is also of form 2 or 3, so $B \in T'$.

3. If C is of form 2 with $n = 0$ then $C = \beta$ and B is $\mathcal{K}(D)$ so is of form 3 and again $B \in T'$.

4. If C is of form 3 with $n = 0$ then C is $x \to E$ for some $E \in T'$ and so $B = E$.

For the remaining cases, we may safely assume that C is $\mathcal{C}_\star(C')$ and that $C' \in T'$. Under this assumption, there are more simple cases:

5. If D is of form 1 (i.e. $D = \alpha$) then the result of unifying D with $C' \to z$ is $C' \to ((C' \to y) \to y)$ where y is not in C', but in that case B is just C, up to rewriting of variables.

6. If D is of form 2 with $n = 0$ then B is $\mathcal{K}(C')$, which is of form 3.

7. If D is of form 3 with $n = 0$ then D is $\mathcal{K}(E)$ for some $E \in T'$, but then $B = E$.

That concludes the special cases. The remaining case has $C = \mathcal{C}^n_\star(\gamma)$ and $D = \mathcal{C}^k_\star(\delta)$ where each of γ and δ is either β or $\mathcal{K}(X)$ for some $X \in T'$. Now the proof proceeds by induction on the product kn. In the base case ($kn = 0$) at least one of k or n is zero. These cases have been treated above. Now assume for induction that $k > 0$ and $n > 0$ and that the result holds for all such pairs of formulae $\mathcal{C}^m_\star(\gamma)$ and $\mathcal{C}^j_\star(\delta)$ with $jm < kn$. The result of the CD inference is that of unifying D with $\mathcal{C}^{n-1}_\star(\gamma) \to z_n$ and detaching whatever gets substituted for z_n in this process. Evidently, since D is $(\mathcal{C}^{k-1}_\star(\delta) \to z_k) \to z_k$, this unifies z_k with z_n and $\mathcal{C}^{n-1}_\star(\gamma)$ with $\mathcal{C}^{k-1}_\star(\delta) \to z_k$. Hence B is also the result of CD with D as major premise and $\mathcal{C}^{n-1}_\star(\gamma)$ as minor, which is in T' by the induction hypothesis. □

Theorem D.7. *SCCD is incomplete.*

Proof. Immediate from Lemmas D.1, D.2, D.3, D.5 and D.6. □

D.3 Remarks

We have seen that the set of support (SOS) strategy is incomplete for condensed detachment problems even when the subtheory initially excluded from the set of support has a model in which the goal theorem is false. This means that it is possible for a prover such as Otter to go wrong on these problems. This incompleteness is not a severe difficulty for Otter, because there is always a way of using the SOS strategy which avoids any failure: at worst, it is possible to put all assumptions except maybe for the detachment clause itself into the set of support, thus guaranteeing completeness at the expense of some efficiency. For SCOTT, however, the situation is different. Because it uses truth values in a model, in effect it constantly re-computes the boundary between axioms and set of support, and because it chooses models dynamically, it sets this boundary aggressively, disallowing as many potential inferences as it can. We want this behaviour, because we want to extract as much guidance as possible from the chosen model, but it is precisely this feature which destroys completeness. Putting everything into the set of support by hand initially avails us nothing if the first thing SCOTT does is to discover the biconditional model which amounts to removing α from the set of support and thus losing the proof.

The situation regarding condensed detachment is but an illustrative example of a much more general problem facing combined systems. Semantic restrictions imposed on proofs as a result of modelling subsets of the formulae derived may, and perhaps typically will, conflict with other constraining mechanisms imposed by the proof strategy. There is no easy general way to control such conflicts except by confining the prover to the use of virtually undirected methods of inference such as crude binary resolution which are not powerful enough for "real" proof search. There are ways out of incompleteness, of course, such as by diluting the semantically directed strategies. Otter, for example, allows the user to decide that, say, nine clauses out of ten will be subject to the guiding strategy but the tenth will be taken from an unregulated breadth-first search just in case it is a bad idea to be over-zealous. These ways out are not really what we want, however. We want to be able to trust our methods, not to adopt an attitude of limiting the damage by employing them less than fully.

A more satisfying way to keep completeness without sacrificing too much of the power of restriction strategies for resolution would be to devise an Otter-like prover using full semantic resolution, rather than just model resolution, with respect to dynamically updated models. Semantic resolution as defined by Slagle is like hyper-resolution in being based on a nucleus and satellites, each satellite picking up a single literal in the nucleus to form a clash. The semantic constraint is that all satellites and the resolvant are required to be false in the guiding model. Whether the clauses involved are positive, negative or mixed is not significant. This form of inference would remain refutation complete, but it is not easy to see how there could be a very fast test for the truth value of the resolvant in the general case.

Another possibility, so far uninvestigated, is to consider ways of detecting incompleteness by some kind of analysis of the prover's behaviour. The particular cases in which incompleteness strikes may not form a decidable set, so methods for detecting them should be expected to be partial. However, there may be insights as well as performance gains to be had from the pursuit of such methods.

Other open lines of research in model-guided theorem proving of the type considered here include the following:

- Generalise the results of the present appendix and understand better the issues of completeness and incompleteness for semantically guided proof search. In particular, characterise classes of problems for which SCCD is complete or for which it is incomplete.

- Implement and investigate other systems in the style of SCOTT. For instance, use other model generation methods such as hill-climbing ones or those based on tableaux or on extensions of resolution. Examine the effects on other kinds of proof search, for instance on the "bottom-up" phase of a prover like SETHEO, and on other types of inference such as equational reasoning.

- Work more on the false preference strategy. This seems to be generally effective with a wide range of inference rules, but there is a lack of firm mathematical results concerning it.

- Explore the possibilities for techniques using multiple models. SCOTT only uses one model at a time, which restricts what it is able to do. False preference in particular offers great possibilities for systems capable of working with many models simultaneously. Competitive parallel proof search in the manner of [70] is another technique obviously suited to guidance by multiple models.

Finally, it must be stressed that the project of harnessing semantic information and putting it in the service of theorem provers is important. The results of this appendix are negative for the research program, certainly, but must not be seen as destructive of it.[9]

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